Math 104 Notes

Kanyes Thaker

Koji Shimizu

Introduction

These are my notes for MATH104 at UC Berkeley - I've excluded the topology introduction here (see my topology notes for those). $\mathbb{N} = \{1, 2, 3, 4, ...\}$ is the set of **natural numbers**.
$\mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$ is the set of **integers**.
$\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$.
$\mathbb{R}$ = the set of **real numbers**.

Some basic definitions

Some important tools in the realm of analysis are the inequality: $x < y$ and absolute value: $|x|$

Absolute value: for $x \in \mathbb{R}$, $|x|$ is the distance from x to 0.

The issue with this definition, upon further observation, is that the word *distance* can have several definitions in different contexts. We will see later that we use the definition of absolute value to give meaning to the intuitive concept of *distance*.

Absolute Value (revised): for $x \in \mathbb{R}$, $|x| = \sqrt{x^2}$

Absolute Value (revised again): for $x \in \mathbb{R}$, $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

We can then say that the *distance* from x to y is $|x - y|$.

Review of Proofs

(1) Prove $1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$
Proof: We can proceed through induction. For $n = 1$, it is immediately clear that $\frac{1\times 2\times 3}{6} = 1$. Now assume that this statement is true for $n = m$, i.e. $\sum_{i=1}^{m} i^2 = \frac{m(m+1)(2m+1)}{6}$. We proceed by induction. $\sum_{i=1}^{m+1} i^2 = \sum_{i=1}^{m} i^2 + (m+1)^2 = \frac{m(m+1)(2m+1)}{6} + (m+1)^2 = \frac{m(m+1)(2m+1)}{6} + \frac{6(m+1)^2}{6} = \frac{m(m+1)(2m+1)+6m^2+12m+6}{6} = \frac{(m+1)(m+2)(2m+3)}{6}$. ■

(2) Prove $\sqrt{7}$ is irrational
Proof: Assume the contrary, that $\sqrt{7} \in \mathbb{Q}$. Then we can express $\sqrt{7}$ as $\frac{m}{n}$, where $m, n \in \mathbb{Z}$ and m and n are coprime. This implies that $\frac{m^2}{n^2} = 7$, and can be written as $m^2 = 7n^2$. Since the LHS and RHS are equivalent, and since 7 is prime (i.e. for prime p, $p|ab \implies p|a \vee p|b$), we know that for some $k \in \mathbb{R}$, $m = 7k \implies 7n^2 = m^2 = 49k^2$. Simplifying, we then see that $n^2 = 7k^2$. By analagous logic to the previous argument, we see that n must also be divisible by 7. This contradicts our initial assertion that m and n are coprime and therefore $\sqrt{7}$ cannot be rational. ■

(3) Prove the triangle inequality, namely that $|x + y| \leq |x| + |y|$
Proof: If we accept the definition of absolute value to be $|x| = \max\{x, -x\}$, the proof proceeds as follows. $|x| + |y| = \max\{x, -x\} + \max\{y, -y\} = \max\{x + y, -x + y, x - y, -x - y\} = \max\{|x + y|, |x - y|\} \geq |x + y|$, since addition distributes across max. ■

The Completeness Axiom

Properties of Absolute Value

$$-|x| \le x \le |x|,\ |x| \le a \text{ if } x \le a \text{ and } -x \le a$$

The Completeness Axiom

Key Question: *What are the differences between rational numbers ($\mathbb{Q}$) and the reals ($\mathbb{R}$)?*
Intuitively, we can say that $\mathbb{Q}$ has many "gaps" in its set while $\mathbb{R}$ is continuous.

Notation: For $a < b : a, b \in \mathbb{R}$, $(a,b) = \{x \in \mathbb{R} : a < x < b\}$ and $[a,b] = \{x \in \mathbb{R} : a \le x \le b\}$.

Definition: Let $\mathcal{S} \in \mathbb{R}$ be a nonempty subset with $m \in \mathbb{R}$.
1) m is the *maximum* of $\mathcal{S}$, i.e. $m = \max\{\mathcal{S}\}$, if $s \le m \forall s \in \mathcal{S}$ and $m \in \mathcal{S}$.
2) m is the *minimum* of $\mathcal{S}$, i.e. $m = \min\{\mathcal{S}\}$ if $s \ge m \forall s \in \mathcal{S}$ and $m \in \mathcal{S}$.

The issue with the ideas of max and min is that they only tell us whether or not a set has a maximum or minimum element, when in reality we're far more interested in what these sets are *bounded by.*

Definition: Define a nonempty $\mathcal{S} \subseteq \mathbb{R}$, $m \in \mathbb{R}$.
1) m is an *upper bound* of $\mathcal{S}$ if $s \le m \forall s \in \mathcal{S}$. $\mathcal{S}$ is *bounded above.*
2) m is a *lower bound* of $\mathcal{S}$ if $s \ge m \forall s \in \mathcal{S}$. $\mathcal{S}$ is *bounded below.*
3) If $\mathcal{S}$ is both bounded above and bounded below, we simply say that $\mathcal{S}$ is *bounded.*
4) m is the *supremum* of $\mathcal{S}$ ($m = \sup \mathcal{S}$) if m is the *smallest upper bound.*
5) m is the *infimum* of $\mathcal{S}$ ($m = \inf \mathcal{S}$) if m is the *largest lower bound.*

This gives us our full **completeness axiom** as follows:

Any nonempty subset of $\mathbb{R}$ that is bounded above admits supremum.

Note: Consider $\mathcal{S} = \{x \in \mathbb{Q} : x^2 < 2\} = (-\sqrt{2}, \sqrt{2}) \cup \mathbb{Q}$ as a subset of $\mathbb{Q}$. Observe that $\mathcal{S} \ne \emptyset$. $\mathcal{S}$ has an upper bound of 100, but $\sup \mathcal{S}$ doesn't exist as a rational number.

Archimedian Property

For each $a > 0$, and for each $b > 0$, $a, b \in \mathbb{R}$, $\exists\ n \in \mathbb{N} : na > b$.

The Denseness of $\mathbb{Q}$

Any real number can be approximated by a rational number that is infinitely close to the value of the real (but not exactly equal). This gives rise to the **denseness of** $\mathbb{Q}$, which states that $\forall a < b$, $a, b \in \mathbb{R}$, $\exists\ r \in \mathbb{Q} : a < r < b$.

Limits of Sequences

A **sequence** is a set whose domain is $\{n \in \mathbb{Z} : n \geq m\}$ where m is either 1 or 0, i.e. a set of values who can be indexed as a function of the positive integers or naturals. They can be denoted as $(s_n)_{n=m}^{\infty}$.

The **limit** of a sequence (s_n) is a real number that the values of the sequence approach for large n.

Question: What is the limit of $a_n = (-1)^n$?

The issue with this question is that for even large n the answer is 1 and for odd large n the answer is -1. We need a precise definition of limits to get a good answer here.

Definition: A sequence (s_n) of real numbers *converges* to $s \in \mathbb{R}$ if for each $\epsilon > 0$, $\exists\ N : n > N \implies |s_n - s| < \epsilon$. If (s_n) *does* converge, we write $\lim_{n\to\infty} s(n) = s$. The number s is then the limit of the sequence. A sequence that doesn't converge *diverges*.

Some Important Limits:

1. $\lim \frac{1}{n^2} = 0$
2. The sequence (a_n) where $a_n = (-1)^n$ does not converge.
3. The sequence $\cos(\frac{n\pi}{3})$ does not converge.
4. The sequence $n^{\frac{1}{n}}$ appears to converge to 1.
5. The sequence (b_n) where $b_n = (1 + \frac{1}{n})^n$ converges to e.

It is important to note as well that limits are *unique*. This means that if $\lim s_n = s$ and $\lim s_n = t$ then $s = t$. This means that if a limit exists for a function, then it is the only limit (limits can't change arbitrarily).

Proof: Consider $\epsilon > 0$. By the definition of limit, $\exists\ N_1 : n > N \implies |s_n - s| < \frac{\epsilon}{2}$. Likewise, for the other limit t, $\exists\ N_2 : n > N_2 \implies |s_n - t| < \frac{\epsilon}{2}$. For $N = \max\{N_1, N_2\}$, the Triangle Inequality tells us $|s - t| = |s - (s_n + s_n) - t| = |(s - s_n) + (s_n - t)| \leq |s - s_n| + |s_n - t| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. ■

A Discussion about Proofs

Here are some examples of rigorous mathematical proofs of the limits from the last section. We care about the *formal proof*, where we can show $p \implies q$ directly using predetermined definitions of ϵ and N. Although in the final proof it's not always obvious how we got those values, we can informally derive those values and work backwards using our formal proof.

Example 1: Prove $\lim \frac{1}{n^2} = 0$
Proof: Our task begins with setting some $\epsilon > 0$ and ultimately ending with $|\frac{1}{n^2} - 0| < \epsilon$. In between, we should be trying to find some N where $n > N \implies |\frac{1}{n^2} - 0| < \epsilon$. In these cases, we will often try and work backwards from the RHS so long as each step is reversible along the way. Here, we can multiply both sides of the inequality by n^2 and divide both sides by ϵ. We then get $\frac{1}{\epsilon} < n^2$ or $n > \frac{1}{\sqrt{\epsilon}}$. If our steps are reversible (which they are), we see that $n > \frac{1}{\epsilon} \implies |\frac{1}{n^2} - 0| < \epsilon$, suggesting we set $N = \frac{1}{\epsilon}$. ■
Proof: (Formal). Let $\epsilon > 0$. Let $N = \frac{1}{\sqrt{\epsilon}}$. Then $n > N \implies n > \frac{1}{\sqrt{\epsilon}} \implies n^2 > \frac{1}{\epsilon} \implies \epsilon > \frac{1}{n^2}$. Therefore $n > N \implies |\frac{1}{n^2} - 0| < \epsilon$. This proves that $\lim \frac{1}{n^2} = 0$. ■

Example 2: Prove $\lim \frac{3n+1}{7n-4} = \frac{3}{7}$
Proof: Again, begin with $\epsilon > 0$, i.e. we are trying to find the minimum $n : |\frac{3n+1}{7n-4} - \frac{3}{7}| < \epsilon$. This can be simplified to $|\frac{21n+7-21n+12}{7(7n-4)}| = |\frac{19}{7(7n-4)}| < \epsilon$. Since the denominator must be positive for $n \in \mathbb{N}$, we can remove the absolute value and solve for n, which gives us $\frac{19}{49\epsilon} + \frac{4}{7} < n$. Since our steps are reversible, we can set $N = \frac{19}{49\epsilon} + \frac{4}{7}$ or any number larger than that quantity. ■
Proof: (Formal). Let $\epsilon > 0$. Let $N = \frac{19}{49\epsilon} + \frac{4}{7}$. Then $n > N \implies n > \frac{19}{49\epsilon} + \frac{4}{7}$, therefore $7n > \frac{19}{7} + 4 \implies \frac{19}{7(7n+4} < \epsilon \implies |\frac{3n+1}{7n-4} - \frac{3}{7}| < \epsilon$, therefore $\lim \frac{3n+1}{7n-4} = \frac{3}{7}$. ■

Example 3: Prove $\lim \frac{4n^3+3n}{n^3-6} = 4$
Proof: For each $\epsilon > 0$, we need to determine the minimum $n : |\frac{4n^3+3n}{n^3-6} - 4| < \epsilon$. Since $n > 1$, we can drop the absolute values. However, in this case it's very difficult to solve for n in terms of ϵ. Instead, we can find an *estimate*. We can do this by realizing that $\frac{4n^3+3n}{n^3-6}$ can be simplified to some constant times $\frac{1}{n^2}$. We can find an upper bound on the numerator and a lower bound on the denominator to find this constant. Realize that for $n > 1$, $3n + 24 \leq 27n$ and that $n^3 - 6 \geq \frac{n^3}{2}$ for large n. We then get $\frac{54}{n} < \epsilon$ or $n > \sqrt{\frac{54}{\epsilon}}$. ■
Proof: (Formal). Let $\epsilon > 0$ and $N = \max\{2, \sqrt{\frac{54}{\epsilon}}\}$. Then $n > N \implies n > \sqrt{\frac{54}{\epsilon}}$, hence $\frac{27n}{n^3/2} < \epsilon$. Since $n > 2$, we know that $\frac{n^3}{2} \leq n^3 - 6$ and also $27n > 3n + 24$. Thus $n > N \implies |\frac{4n^3+3n}{n^3-6} - 4| < \epsilon$ as desired. ■

Limit Theorems for Sequences

Theorem: *Convergent sequences are bounded.*

Proof: Let (s_n) be a convergent sequence, and let $s = \lim s_n$. Let $\epsilon = 1$, and take $N \in \mathbb{N} : n > n \implies |s_n - s| < 1$. From the traingle inequality we see $n > N \implies |s_n| < |s| + 1$. $M := \max\{|s| + 1, |s_1|, ..., |s_n|\}$. Then $|s_n| \leq M \forall n \in \mathbb{N}$, so (s_n) is bounded. ■

Theorem: *If the sequence (s_n) converges to s and $k \in \mathbb{R}$, then the sequence (ks_n) converges to ks. That is,* $\lim(ks_n) = k \cdot \lim s_n$.

Proof: Assume $k \neq 0$, since this result is trivial. Let $\epsilon > 0$ and note that we need to show that $|ks_n - ks| < \epsilon$ for large n. Since $\lim s_n = s$, $\exists\ N : n > N \implies |s_n - s| < \frac{\epsilon}{|k|}$. Then, $n > N \implies |ks_n - ks| < \epsilon$. ■

Theorem: *If $(s_n$ converges to s (t_n) converges to t, then $(s_n + t_n)$ converges to $s + t$. That is,* $\lim(s_n + t_n) = \lim s_n + \lim t_n$.

Proof: Let $\epsilon > 0$; we need to show $|s_n + t_n - (s + t)| < \epsilon$ for large n. From the triangle inequality we get $|s_n - s| + |t_n - t| \geq |s_n + t_n - (s + t)|$. Then $\exists$ $N_1 : n > N_1 \implies |s_n - s| < \frac{\epsilon}{2}$ and $\exists\ N_2 : n > N_2 \implies |t_n - t| < \frac{\epsilon}{2}$. Then if $N = \max\{N_1, N_2\}$, $n > N \implies |s_n + t_n - (s + t)| \leq |s_n - s| + |t_n - t| < \epsilon$. ■

Theorem: *If (s_n) converges to s and (t_n) converges to t, then $(s_n t_n)$ converges to st. That is,* $\lim(s_n t_n) = (\lim s_n)(\lim t_n)$.

Proof: Let $\epsilon > 0$. $\exists\ M > 0 : |s_n| \leq M \forall n$. Since $\lim t_n = t$ $\exists\ N_1 : n > N_1 \implies |t_n - t| < \frac{\epsilon}{2M}$. Since $\lim s_n = s \implies \exists\ N_2 : n > N_2 \implies |s_n - s| < \frac{\epsilon}{2(|t|+1)}$. Now if $N = \max\{N_1, N_2\}, n > N \implies |s_n t_n - st| \leq |s_n| \cdot |t_n - t| + |t| \cdot |s_n - s| \leq M \cdot \frac{\epsilon}{2M} + |t| \cdot \frac{\epsilon}{2(|t|+1)} < \epsilon$. ■

Theorem: *If (s_n) converges to s, if $s_n \neq 0$ for all n, and if $s \neq 0$, then $(1/s_n)$ converges to $1/s$.*

Proof: Let $\epsilon > 0$. We know that for a convergent sequence $\exists\, m > 0 : |s_n| \geq m \forall n$. Since $\lim s_n = s$, $\exists\ N : n > N \implies |s - s_n| < \epsilon \cdot m|s|$. Then $n > N \implies |\frac{1}{s_n} - \frac{1}{s}| = \frac{|s - s_n|}{|s_n s|} \leq \frac{|s - s_n|}{m|s|} < \epsilon$. ■

Theorem: *Suppose (s_n) converges to s and (t_n) converges to t. Then* $\lim \frac{t_n}{s_n}$ *converges to* $\frac{t}{s}$.
Theorem: $\lim(\frac{1}{n^p}) = 0$.
Theorem: $\lim a^n = 0$ if $|a| < 1$.
Theorem: $\lim n^{1/n} = 1$.
Theorem: $\lim a^{1/n} = 1$ if $a > 0$.

Definition: For a sequence (s_n), we write $\lim s_n = +\infty$ provided for each $M > 0$ there is a number N such that $n > N \implies s_n > M$. In this case we say the sequence *diverges* to $+\infty$. We write $\lim s_n = -\infty$ provided for each $M < 0$ there is a number N such that $n > N \implies s_n < M$.

Theorem: *Let* (s_n) *and* (t_n) *be sequences such that* $\lim s_n = +\infty$ *and* $\lim t_n > 0$. *Then* $\lim s_n t_n = +\infty$.

Proof: Let $M > 0$. Select a real number m such that $0 < m < \lim t_n$. It is clear that $\exists N_1 : n > N_1 \implies t_n > m$. Since $\lim s_n = +\infty$, $\exists N_2 : n > N_2 \implies s_n > \frac{M}{m}$. Put $N = \max\{N_1, N_2\}$. Then $n > N \implies s_n t_n > \frac{M}{m} \cdot m = M$. ∎

Theorem: *For a sequence* (s_n) *of positive real numbers, we have* $\lim s_n = +\infty$ *if and only if* $\lim \frac{1}{s_n} = 0$.

Monotone Sequences and Cauchy Sequences

So far, we've dealt with scenarios where we can check whether a certain sequences converges to a limit. However, in reality we're often faced with scenarios where we don't know what the limit is beforehand.

Definition: A sequence (s_n) of real numbers is called an *increasing sequence* if $s_n \leq s_{n+1}\ \forall n$, and (s_n) is called a *decreasing sequence* if $s_n \geq s_{n+1}\ \forall n$. Inductively, this means that for an increasing sequence $m > n \implies s_m \geq s_n$, and a symmetric conclusion can be drawn for decreasing sequences. A sequence that is increasing or decreasing can be called *monotone* or *monotonic*.

Theorem: *All bounded monotone sequences converge.*

Proof: Let (s_n) be a bounded sequence. Let $\mathcal{S} := \{s_n : n \in \mathbb{N}\}$, and $u := \sup \mathcal{S}$. Since $\mathcal{S}$ is bounded, $u \in \mathbb{R}$. We will now prove that $\lim s_n = u$.
Let $\epsilon > 0$. $u - \epsilon$ is not an upper bound for $\mathcal{S}$, so we know that $\exists\ N : n > N \implies s_N > u - \epsilon$. By the definition of monotonic sequences, $n > N \implies s_N \leq s_n$. Of course, $s_n \leq u$ for all n, so $n > N \implies u - \epsilon < s_n \leq u$, implying that $|s_n - u| < \epsilon$. Therefore $\lim s_n = u$. ■

Decimals
We have to be careful with how we represent real numbers. Real numbers are simply decimal expansions, but there are certain subtleties concerning their notation (i.e. different decimal expansions can represent the same real number). For now, let's only look at nonnegative real numbers.

Suppose we have a nonnegative real number $K.d_1d_2d_3...d_n$, where $K \in \mathbb{Z} \geq 0$, and $d_j \in \{0,1,2,3,4,5,6,7,8,9\}$. Then let $s_n = K + \frac{d_1}{10} + \frac{d_2}{10^2} + ... + \frac{d_n}{10^n}$. Then (s_n) is an increasing sequence of real numbers, and is bounded (by $K+1$). Therefore, by the aforementioned theorem, this number must converge to a real number $K.d_1d_2d_3...$. For example, the real number 3.3333... can be represented as $\lim_{n \to \infty}(d + \frac{3}{10} + ... + \frac{3}{10^n})$.

We can use the fact that for $|r| < 1$, the sum of $\lim_{n\to\infty} a(1 + r + r^2 + ... + r_n) = \frac{a}{1-r}$ to show that this limit is $\frac{10}{3}$, as expected. However, looking at the sequence $\lim(\frac{9}{10} + ... + \frac{9}{10^n}) = 0.9999999... = 1$. 1.00000... and 0.999999... are the same number!

Theorem:
i. If (s_n) is an unbounded increasing sequence, then $\lim s_n = +\infty$.
ii. If (s_n) is an unbounded decreasing sequence, then $\lim s_n = -\infty$.

Proof: i. Let (s_n) be an unbounded sequence increasing sequence. Let $M > 0$. Since the set $\{s_n : n \in \mathbb{N}\}$ has a lower bound in s_1, it must be unbounded above. Then for some $N \in \mathbb{N}$, we have $s_N > M$. It's immediately obvious that $n > N \implies s_n \geq s_N > M$. Then $\lim s_n = +\infty$. A symmetric proof follows for *ii.* ■

Corrolary: *If s_n is monotone, then the sequence either converges or diverges to either $+\infty$ or $-\infty$. Therefore $\lim s_n$ is always meaningful for monotone sequences.*

Definition: Let (s_n) be a sequence in $\mathbb{R}$. Define the following:

1) $\limsup s_n = \lim_{N\to\infty} \sup\{s_n : n > N\}$.

2) $\liminf s_n = \lim_{N\to\infty} \inf\{s_n : n > N\}$.

Note: These definitions make no mention of whether or not s_n is bounded. We use the following conventions: if (s_n) is not bounded above, then $\limsup s_n = +\infty$, and if it is not bounded below then $\liminf s_n = -\infty$.

A clarification is necessary here: $\limsup s_n$ doesn't have to be the same as $\sup\{s_n : n \in \mathbb{N}\}$, but $\limsup s_n \leq \sup\{s_n : n \in \mathbb{N}\}$. Some elements of s_n can be much, much larger than $\limsup s_n$ – $\limsup s_n$ is the smallest value of s_n that an infinite number of s_n's can get close to. If this seems confusing, that's because it is, but it will be clarified in later sections. For now, we need to say that $\lim s_n$ exists iff $\liminf s_n = \limsup s_n$.

Theorem: *Let (s_n) be a sequence in $\mathbb{R}$.*

i. If $\lim s_n$ is defined [a real number, $\pm\infty$], then $\liminf s_n = \limsup s_n = \lim s_n$.

ii. If $\limsup s_n = \liminf s_n$, then $\lim s_n$ is defined and $\lim s_n = \limsup s_n = \liminf s_n$.

If s_n converges, then $\liminf s_n = \limsup s_n$ by the above theorem, so for large N the numbers $\sup\{s_n : n > N\}$ and $\inf\{s_n : n > N\}$ are close together. This means that all numbers in the set $\{s_n : n > N\}$ are close together. This concept will be **very important** later in this course.

Definition: A sequence (s_n) of real numbers is a *Cauchy sequence* if for each $\epsilon > 0$, $\exists\ N : m, n > N \implies |s_n - s_m| < \epsilon$.

Lemma: *Convergent sequences are Cauchy sequences.*

Proof: Suppose $\lim s_n = s$. The idea is that since the terms for large n are close to s, they are also close to each other. In other words, letting $\epsilon > 0$, $n > N \implies |s_n - s| < \frac{\epsilon}{2}$ and $m > N \implies |s_m - s| < \frac{\epsilon}{2}$, so $m, n > N \implies |s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s - s_m| < \epsilon$ ■

Lemma: *Cauchy sequences are bounded.*

Proof: Apply an analgous proof as the 1st Theorem from the previous lecture with the definition of Cauchy sequences. ■

Theorem: *A sequence is a convergent sequence if and only if it is a Cauchy sequence.*

Subsequences

Definition: Suppose (s_n) is a sequence. A *subsequence* of the sequence is a sequence of the form (t_k), where for each $k \exists n_k : n_1 < ... < n_k < n_{k+1} < ...$ and $t_k = s_{n_k}$. In essence, a subsequence is a subset of (s_n) selected in order.

A more formal definition: view the sequcne (s_n) as a function s with domain $\mathbb{N}$. For the subset $\{n_1, ...\}$, there is another function σ where $\sigma(k) = n_k$, i.e. it "selects" a subset of $\mathbb{N}$ in order. Then the subsequence of s corresponding to the function σ is $t = s \circ \sigma$. That is,

$$t_k = t(k) = s \circ \sigma(k) = s(\sigma(k)) = s(n_k) = s_{n_k}, k \in \mathbb{N}.$$

Example 1: Let $s_n = (-1)^n n^2$. Then $(s_n) = (-1, 4, -9, 16, -25, 36, ...)$. Take the subsequence $(4, 16, 36, 64...)$. The subsequence is then s_{n_k}, where $n_k = 2k$, so $s_{n_k} = 4k^2$ and the selection function is $\sigma = 2k$.

Theorem: *Let (s_n) be a sequence.*

i. If $t \in \mathbb{R}$, $\exists$ a subsequence of (s_n) converging to t iff the set $\{n \in \mathbb{N} : |s_n - t| < \epsilon\}$ is infinite for all $\epsilon > 0$.
ii. If the sequence (s_n) is unbounded above, it has a subsequence with limit $+\infty$.
iii. If the sequence (s_n) is unbounded below, it has a subsequence with limit $-\infty$.

Example 2: We can show that $\mathbb{Q}$ can be represented as a sequence (r_n). In set theory, we could use this to say that $\mathbb{Q}$ is countable. There are an infinite number of rationals in the interval $(a - \epsilon, a + \epsilon)$ for any $\epsilon > 0$, so based on the above theorem we can say that for any real number a, there exists a subsequence of $\mathbb{Q}$ converging to a. We referred to this much earlier when we discussed the "denseness of $\mathbb{Q}$."

Theorem: *If the sequence (s_n) converges, then every subsequence converges to the same limit.*
Proof: Let (s_{n_k}) be a subsequence of (s_n). $n_k \geq k$, which can be easily shown by induction. Let $s = \lim s_n$, $\epsilon > 0$. Then $\exists\ N \in \mathbb{N} : n > N \implies |s_n - s| < \epsilon$. Now $k > N \implies n_k > N$, so $|s_{n_k} - s| < \epsilon$, so $\lim s_{n_k} = s$. ■
Theorem: *Every sequence (s_n) has a monotonic subsequence.*

The Bolzano-Weierstrauss Theorem

Theorem: *Every bounded sequence has a convergent subsequence.*
Definition: Let (s_n) be a sequence in $\mathbb{R}$. A *subsequential limit* is any real number or $\pm\infty$ that is the limit of some subsequence (s_{n_k}).
Theorem: *Let (s_n) be a sequence. There exists a monotonic subsequence whose limit is $\limsup s_n$ and there exists a monotonic subsequence whose limit is $\lim \in s_n$.*

Theorem: *Let (s_n) be any sequence in $\mathbb{R}$, and let $\mathcal{S}$ denote the set of subsequential limits of (s_n).*

i. $\mathcal{S}$ is nonempty.

ii. $\sup \mathcal{S} = \limsup s_n$ and $\inf \mathcal{S} = \liminf s_n$.

iii. $\lim s_n$ exists iff $\mathcal{S}$ has exactly one element.

Theorem: *Let $\mathcal{S}$ denote the set of subsequential limits of a sequence (s_n). Suppose (t_n) is a sequence in $\mathcal{S} \cup \mathbb{R}$ and that $t = \lim t_n$. Then t belongs to $\mathcal{S}$.*

lim sup's and lim inf's

Let (s_n) be any sequence of real numbers. Let $\mathcal{S}$ be the set of subsequential limits of (s_n). Recall that

$$\limsup s_n = \lim \sup\{s_n : n > N\} = \sup \mathcal{S}$$

$$\liminf s_n = \inf \sup\{s_n : n > N\} = \inf \mathcal{S}$$

Theorem: *If (s_n) converges to a positive $s \in \mathbb{R}$ and (t_n) is any sequence, then $\limsup s_n t_n = s \limsup t_n$.*

Theorem: *Let (s_n) be any sequence of nonzero real numbers. Then*

$$\liminf |\frac{s_{n+1}}{s_n}| \leq \liminf |s_n|^{1/n} \leq \limsup |s_n|^{1/n} \leq \limsup |\frac{s_{n+1}}{s_n}|$$

Corrolary: *If $\lim |\frac{s_{n+1}}{s_n}|$ exists and equals L, then $\lim |s_n|^{1/n}$ exists and equals L.*

Some Topological Concepts in Metric Subspaces

Definition: Let $\mathcal{S}$ be a set and let d be a function that is defined for all pairs (x, y) of elements of $\mathcal{S}$ satisfying

D1. $d(x, x) = 0\ \forall x \in \mathcal{S}$ and $d(x, y) > 0$ for distinct $x, y \in \mathcal{S}$.

D2. $d(x, y) = d(y, x)\ \forall x, y \in \mathcal{S}$.

D3. $d(x, z) \leq d(x, y) + d(y, z)\ \forall x, y \in \mathcal{S}$.

We call d a *metric* or *distance function* on $\mathcal{S}$. $\mathcal{S}$ is a metric space is a set combined with the metric on it. More formally, the metric space is $(\mathcal{S}, d)$ since $\mathcal{S}$ may have multiple metrics.

Definition: A sequence (s_n) in a metric space $(\mathcal{S}, d)$ converges to s in $\mathcal{S}$ if $\lim_{n\to\infty} d(s_n, s) = 0$. A sequence (s_n) in $\mathcal{S}$ is *Cauchy* if

$$\forall \epsilon > 0 \exists N : m, n > N \implies d(s_m, s_n) < \epsilon.$$

A metric space is *complete* if every Cauchy sequence in $\mathcal{S}$ converges to an element of $\mathcal{S}$.We use $(\boldsymbol{x}^{(n)})$ to refer to a sequence instead of $(\boldsymbol{x}_n)$.

Lemma: *A sequence $(\boldsymbol{x}^{(n)})$ in $\mathbb{R}^k$ converges iff $\forall j = 1, 2, ..., k$ the sequence $(x_j^{(n)})$ converges in $\mathbb{R}$. A sequence $(\boldsymbol{x}^{(n)})$ in $\mathbb{R}^k$ is a Cauchy sequence iff each $(x_j^{(n)})$ is Cauchy in $\mathbb{R}$.*

Series

Summation Notation

We use the symbol $\sum_{k=m}^{n} a_k$ as a shorthand for the summation $a_m + a_{m+1} + ... + a_{n-1} + a_n$. We use the notation $\sum_{k=m}^{\infty} a_k$ to describe an *infinite sum*, which we assign meaning to below.

Infinite Series

To give meaning to $\sum_{k=m}^{\infty}$ we consider the sequence $(s_n)_{n=m}^{\infty}$ of partial sums, where

$$s_n = a_m + a_{m+1} + ... + a_{n-1} + a_n = \sum_{k=m}^{n} a_k$$

The infinite series $\sum_{k=m}^{\infty}$ converges provided the sequence (s_n) of partial sums converges to $S \in \mathbb{R}$, where S is also the limit of the series. In other words,

$$\sum_{n=m}^{\infty} a_n = S \iff \lim s_n = S \iff \lim_{n \to \infty} \left(\sum_{k=m}^{n} a_k = S \right)$$

We say a sequence *diverges* to $+\infty$ if its sequence of partial sums has a limit of $+\infty$. A symmetric definition follows for $-\infty$. If all the terms a_n of the series $\sum a_n$ are nonnegative, then the sequence of partial sums is increasing and must either converge or diverge to $+\infty$. We can use this fact to state that $\sum |a_n|$ is meaningful for any (a_n). If $\sum a_n$ converges, we call the series *absolutely convergent*. All absolutely convergent series are convergent.

Example 1: A series of the form $\sum ar^n$. For $r \neq 1$, the partial sums s_n are

$$\sum_{k=0}^{n} ar^k = a\frac{1 - r^{n+1}}{1 - r}$$

If $|r| < 1$, then $\lim_{n \to \infty} r^{n+1} = 0$, so $\lim_{n \to \infty} s_n = \frac{a}{1-r}$. Then

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}, |a| < 1$$

Consider a fixed positive $p \in \mathbb{R}^n$. Then

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if and only if } p > 1.$$

Definition: We say a series $\sum a_n$ satisfies the *Cauchy criterion* if its sequence of partial sums (s_n) is a Cauchy sequence:

$$\forall \epsilon > 0 \exists N \in \mathbb{N} : m, n \in \mathbb{N} > N \implies |s_n - s_m| < \epsilon$$

Without loss of generality, we can rewrite this as

$$\forall \epsilon > 0 \exists N \in \mathbb{N} : n \geq m > N \implies \left| \sum_{k=m}^{n} a_k \right| < \epsilon$$

Theorem: *A series converges iff it satisfies the Cauchy criterion.*
Corrolary: *If a series* $\sum a_n$ *converges, then* $\lim a_n = 0$.

Comparison Test

Let $\sum a_n$ be a series where $a_n \geq 0 \forall n$.
  i. If $\sum a_n$ and $|b_n| \leq a_n \forall n$, then $\sum b_n$ converges.
  ii. If $\sum a_n = +\infty$ and $b_n \geq a_n \forall n$, then $\sum b_n = +\infty$.

Corrolary: *Absolutely convergent series are convergent.*

Proof: Suppose $\sum b_n$ is absolutely convergent. Then $\sum a_n$ converges where $a_n = |b_n| \forall n$. Then trivially $|b_n| \leq a_n$ so $\sum b_n$ converges. ■

Ratio Test

A series $\sum a_n$ of nonzero terms
  i. Converges absolutely if $\limsup |a_{n+1}/a_n| < 1$
  ii. diverges if $\liminf |a_{n+1}/a_n| > 1$
  iii. Else $\liminf |a_{n+1}/a_n| \leq 1 \leq \limsup |a_{n+1}/a_n|$, the test is not valuable.

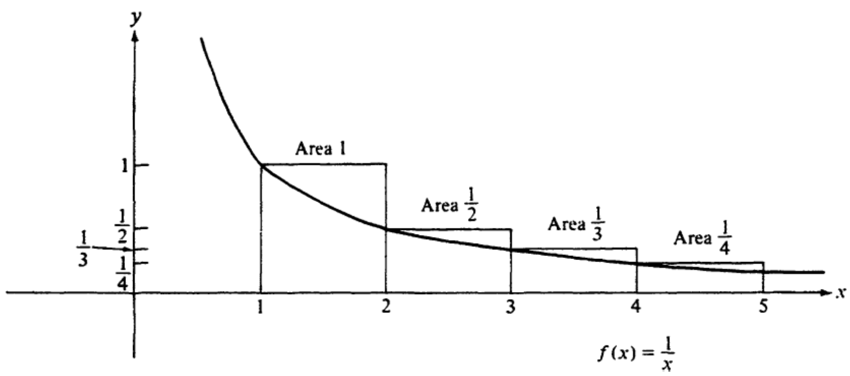
Root Test

Let $\sum a_n$ be a series and let $\alpha = \limsup |a|^{1/n}$. Then $\sum a_n$
  i. Converges absolutely if $\alpha < 1$
  ii. Diverges if $\alpha > 1$.

Alternating Series and Integral Tests

We can check convergence and divergence by comparing the partial sums of a series with common integrals.

Example 1: We show that $\sum \frac{1}{n} = +\infty$. Examine the following diagram:

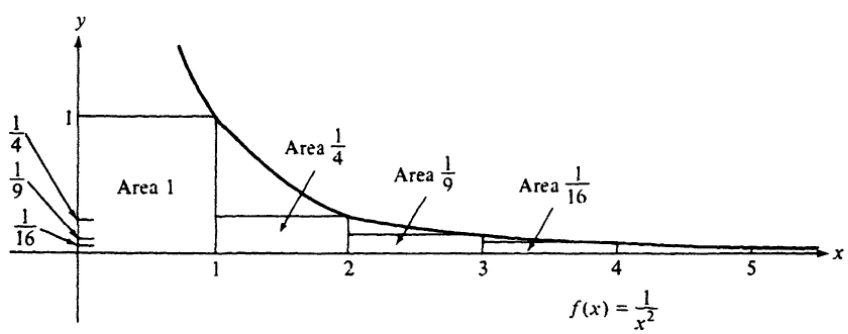


It is evident that for $n \geq 1$, $\sum_{k=1}^{n} \frac{1}{k}$ is the sum of the areas of the first n rectangles of $\frac{1}{k}$. This quantity is greater than the area under the curve from 1 to $n+1$, so we can put a lower bound on this area of

$$\int_{1}^{n+1} \frac{1}{x} dx = \ln(n+1)$$

Since $\lim_{n\to\infty} \ln(n+1) = +\infty$, we can conclude that $\sum \frac{1}{n} = +\infty$.

Example 2: We show $\sum \frac{1}{n^2}$ converges. Examine the following diagram:



From this image, we can see that

$$\sum_{k=1}^{n} \frac{1}{k^2} = \text{sum of the areas of the first n rectangles} \leq 1 + \int_{1}^{n} \frac{1}{x^2} dx = 2 - \frac{1}{n} < 2$$

From this we can see that the sequence of partial sums is increasing and upper bounded by 2.
Integral Test: Use the integral tests if:

1. None of the other test methods apply.
2. All the terms of the series are nonnegative.
3. There is a nice decreasing function f where $f(n) = a_n$ for all n.
4. The integral of f is easy to either calculate or estimate.

The Alternating Series Theorem
Theorem: *If* $a_1 \geq ... \geq a_n$ *and* $\lim a_n = 0$*, then the alternating series* $\sum(-1)^{n+1}a_n$ *converges. Moreover, the partial sums* $s_n = \sum_{k+1}^{n}(-1)^{k+1}a_k$ *satisfy* $|s - s_n| \leq a_n \forall n$.

Continuous Functions

The salient features of a function f are the *set* on which f is defined, known as the domain of f ($\text{dom}(f)$), and the assignment, rule or formula specifying $f(x)$ on each $x \in \text{dom}(f)$.

We work with *real-valued* functions, functions where $\text{dom}(f) \subseteq \mathbb{R}$ and $f(x) \in \mathbb{R} \forall x \in \text{dom}(f)$. We often assume that the function's domain, if not explicitly stated, is the *natural domain*, i.e. the largest $\mathcal{D} \subseteq \mathbb{R}$ where f is well-defined. We begin by defining continuity in terms of sequences along with the usual $\epsilon - \delta$ definition.

Definition: Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. f is continuous at $x_0 \in \text{dom}(f)$ if $\forall (x_n) \in \text{dom}(f)$ converging to x_0, $\lim_n f(x_n) = f(x_0)$. If f is continuous at each point in $\mathcal{S} \subseteq \text{dom}(f)$, we say that f is continuous on $\mathcal{S}$. f is continuous if it is continuous on $\text{dom}(f)$.

Theorem: *Let f be a real-valued function whose domain is $\subseteq \mathbb{R}$. Then f is continuous at $x_0 \in dom(f)$ if and only if*

$$\forall \epsilon > 0 \exists \delta > 0 : x \in dom(f) \vee |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$$

Theorem: *Let f be a real-valued function with $dom(f) \subseteq \mathbb{R}$. If f is continuous at $x_0 \in dom(f)$, then $|f|$ and kf, $k \in \mathbb{R}$, are continuous at x_0.*

Remember that for real-valued functions f and g, the following apply:

1. $(f+g)(x) = f(x) + g(x); \text{dom}(f+g) = \text{dom}(f) \cap \text{dom}(g)$
2. $(fg)(x) = f(x)g(x); \text{dom}(fg) = \text{dom}(f) \cap \text{dom}(g)$
3. $(f/g)(x) = \frac{f(x)}{g(x)}; \text{dom}(f/g) = \text{dom}(f) \cap \{x \in \text{dom}(g) : g(x) \neq 0\}$
4. $f \circ g(x) = f(g(x)); \text{dom}(f \circ g) = \{x \in \text{dom}(g) : g(x) \in \text{dom}(f)\}$
5. $\max(f,g)(x) = \max\{f(x), g(x)\}; \text{dom}(\max(f,g)) = \text{dom}(f) \cap \text{dom}(g)$
6. $\min(f,g)(x) = \min\{f(x), g(x)\}; \text{dom}(\min(f,g)) = \text{dom}(f) \cap \text{dom}(g)$

If f and g are real-valued and continuous at x_0, then $f+g$, fg, and f/g are continuous at x_0 if $g(x_0) \neq 0$ in the third case. If g is continuous at x_0 and f is continuous at $g(x_0)$ then $f \circ g$ is continuous at x_0.

Properties of Continuous Functions

A real valued function is *bounded* if $\{f(x) : x \in \text{dom}(f)\}$ is bounded, i.e. $|f(x)| \leq M \in \mathbb{R} \forall x \in \text{dom}(f)$.

Theorem: *Let f be a continuous real-valued function on a closed interval $[a, b]$. Then f is a bounded function. Additionally, f assumes its minimum nad maximum values on $[a, b]$, i.e.*

$$\exists x_0, y_0 \in [a, b] : f(x_0) \leq f(x) \leq f(y_0) \forall x \in [a, b]$$

Proof: Assume f is unbounded on $[a, b]$. Then for each $n \in \mathbb{N} \exists x_n \in [a, b] : f(x_n) > n$. By the Bolzano-Weierstrass theorem, (x_n) has a subsequence (x_{n_k}) that converges to some $x_0 \in [a, b]$. Since f is continuous at x_0, $\lim(x_{n_k}) = x_0$; however, $\lim(x_{n_k}) = +\infty$ as well. Contradiction, so f is bounded.
Next, let $M = \sup\{f(x) : x \in [a, b]\}$. M is finite. $\forall n \in \mathbb{N}, \exists y_n \in [a, b] : M - \frac{1}{n} < f(y_n) \leq M$. Then $\lim f(y_n) = M$. By Bolzano-Weierstrass, $\exists (y_{n_k})$, a subsequence of (y_n) which converges to $y_0 \in [a, b]$. Since f is continuous at y_0, $\lim f(y_{n_k}) = y_0$. Since $f(y_{n_k})$ is a subsequence of $f(y_n)$, $\lim f(y_{n_k}) = \lim f(y_n) = M$ so $f(y_0) = M$. Therefore f has a maximum at M. We can apply a symmetric argument for $-f$ to achieve the minimum. ■

Intermediate Value Theorem: *If f is a continuous real-valued function on some interval I, then f has the* ***intermediate property*** *on I: Whenever $a, b \in I$, $a < b$ and $f(a) < y < f(b)$ or $f(b) < y < f(b)$, there is some $x \in (a, b) : f(x) = y$.*

Corrolary: *If f is a continuous real-valued function on some interval I, then $f(I) = \{f(x), x \in I\}$ is also an interval or single point.*

Theorem: *Let f be a continuous strictly increasing function on some interval I. Then $f(I)$ is an interval J and f^{-1} is a function with domain J. Then f^{-1} is a continuous strictly increasing function on J.*

Theorem: *Let f be a one-to-one continuous function on an interval I. Then f is either strictly increasing or strictly decreasing.*

Uniform Continuity

Let f be a real-valued function where $\text{dom}(f) \subseteq \mathbb{R}$. Recall that f is continuous on $\mathcal{S} \subseteq \mathbb{R}$ if and only if

$$\forall x_0 \in \mathcal{S}, \epsilon > 0, \exists \delta > 0 : x \in \text{dom}(f), |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$$

It's evident here that the value we choose for δ is dependent on $\epsilon > 0$ and also on our choice of x_0. It would be really useful to know *when* we can find δ solely based on knowledge of $\epsilon > 0$ and $\mathcal{S}$, without having to make rough estimates for each $\boldsymbol{x}_0$. Functions that have this property are called *uniformly continuous.* For clarity, we now refer to x as x and x_0 as y.

Definition: Let f be a real-valued function defined on $\mathcal{S} \in \mathbb{R}$. Then f is *uniformly continuous* on $\mathcal{S}$ if

$$\forall \epsilon > 0 \exists \delta > 0 : x, y \in \mathcal{S}, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Theorem: *If f is continuous on a closed interval $[a, b]$, then f is uniformly continuous on $[a, b]$*

Theorem: *If f is uniformly continuous on a set $\mathcal{S}$ and (s_n) is a Cauchy sequence in $\mathcal{S}$ then $(f(s_n))$ is a Cauchy sequence.*

For the next theorem we use *extensions of functions.* $\tilde{f}$ is an extension of f if $\text{dom}(f) \subseteq \text{dom}(\tilde{f})$ and $f(x) = \tilde{f}(x)\ \forall x \in \text{dom}(f)$.

Theorem: *A real-valued function f on (a, b) is uniformly continuous on (a, b) if and only if it can be extended to a continuous function $\tilde{f}$ on $[a, b]$.*

Theorem: *Let f be a continuous function on an interval I, bounded or unbounded. Let I° be the interval obtained by removing from I any endpoints in I. If f is differentiable on I° and if f' is bounded on I°, then f is uniformly continuous on I.*

Limits of Functions

We've stated that the definition of a function $f(x)$ continuous at a is that the values of $f(x)$ approach $f(a)$, $x \in \text{dom}(f)$. We can easily translate this notion to that of *limits*, i.e. $\lim_{x \to a} f(x) = f(a)$.

Definition: Let $\mathcal{S} \subseteq \mathbb{R}$. Let $a \in \mathbb{R}$ or $\pm\infty$ that is the limit of some sequence in $\mathcal{S}$ and let L be some real number or $\pm\infty$. Then $\lim_{x \to a} f(x) = L$ if

$$f \text{ is a function defined on } \mathcal{S}(1)$$

and

$$\text{for every sequence } (x_n) \text{ in } \mathcal{S} \text{ with limit } a, \ \lim_{n \to \infty} f(x_n) = L(2)$$

Here we define the notation $\lim_{x \to a^{\mathcal{S}}}$ as the limit as x tends to a along $\mathcal{S}$. From here, it is easy to see that a function f is continuous at a in $\text{dom}(f) = \mathcal{S}$ iff $\lim_{x \to a^{\mathcal{S}}} f(x) = f(a)$. Additionally, remember that limits of sequences, should they exist, are *unique*.

Definition:

a. For $a \in \mathbb{R}$ and some function f we write $\lim_{x \to a} f(x) = L$ provided $\lim_{x \to a^{\mathcal{S}}} f(x) = L$ for some $\mathcal{S} = \mathcal{J} \backslash \{a\}$ where $\mathcal{J}$ is an open interval containing a. This limit is called the *two-sided* limit of f at a. This definition allows us to look for a limit even if f is not defined at a. The only time when $f(a) = \lim_{x \to a} f(x)$ is when f is defined on an open interval containing a and $f(x)$ is continuous at a.
b. For $a \in \mathbb{R}$ and a function f we write $\lim_{x \to a^+} f(x) = L$ provided $\lim_{x \to a^{\mathcal{S}}} f(x) = L$ for some open interval $\mathcal{S} = (a, b)$. $\lim_{x \to a^+} f(x)$ is the right-hand limit of f at a. Again f need not be defined at a.
c. For $a \in \mathbb{R}$ and a function f we write $\lim_{x \to a^-} f(x) = L$ provided $\lim_{x \to a^{\mathcal{S}}} f(x) = L$ for some open interval $\mathcal{S} = (c, a)$. $\lim_{x \to a^-} f(x)$ is the left-hand limit of f at a.
d. For a function f we write $\lim_{x \to \infty} f(x) = L$ provided $\lim_{x \to \infty^{\mathcal{S}}} f(x) = L$ for some $\mathcal{S} = (c, \infty)$. Analogously, we write $\lim_{x \to -\infty} f(x) = L$ provided $\lim_{x \to -\infty^{\mathcal{S}}} f(x) = L$ for some $\mathcal{S} = (-\infty, c)$.

Theorem: *Let f_1 and f_2 be functions for which the limits $L_1 = \lim_{x \to a^{\mathcal{S}}} f_1(x)$ and $L_2 = \lim_{x \to a^{\mathcal{S}}} f_2(x)$ exist and are finite. Then*

i. $\lim_{x \to a^{\mathcal{S}}} (f_1 + f_2)(x) = L_1 + L_2$
ii. $\lim_{x \to a^{\mathcal{S}}} (f_1 f_2)(x) = L_1 L_2$
iii. $\lim_{x \to a^{\mathcal{S}}} (f_1 / f_2)(x) = L_1 / L_2$ *provided* $L_2 \neq 0$ *and* $f_2(x) \neq 0, x \in \mathcal{S}$.

The proofs for the above are simple and follow from our limit properties.

Theorem: *Let f be a function for which the limit $L = \lim_{x \to a^{\mathcal{S}}} f(x)$ exists and is finite. If g is a function defined on $\{f(x) : x \in S\} \cup \{L\}$ that is continuous at L, then $\lim_{x \to a^{\mathcal{S}}} g \circ f(x)$ exists and equals $g(L)$.*

Note that it is *essential* that g be continuous at L (for a reason why, reference $f(x) = 1 + x\sin(\frac{\pi}{x})$ and $g(x) = 4$ if $x \neq 1$ and $g(x) = -4$ for $x = 1$, and observe what happens for $(x_n) = \frac{2}{n}$.

Theorem: *Let f be a function defined on a subset $\mathcal{S}$ of $\mathbb{R}$, let a be a real number that is the limit of some sequence in $\mathcal{S}$, and let $L \in \mathbb{R}$. Then $\lim_{x \longrightarrow a^{\mathcal{S}}} f(x) = L$ iff*

$$\forall \epsilon > 0 \exists \delta > 0 : x \in \mathcal{S}, |x - a| < \delta \implies |f(x) - L| < \epsilon.$$

Corrolary: *Let f be a function defined on $\mathcal{J}\backslash\{a\}$ for some open interval $\mathcal{J}$ containing a and let $L \in \mathbb{R}$. Then $\lim_{x \longrightarrow a} f(x) = L$ iff*

$$\forall \epsilon > 0 \exists \delta > 0 : 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$

Corrolary: *Let f be a function defined on (a, b), and let L be a real number. Then $\lim_{x \longrightarrow a^+} f(x) = L$ iff*

$$\forall \epsilon > 0 \exists \delta > 0 : a < x < \delta + a \implies |f(x) - L| < \epsilon.$$

Let's use the above corrolaries to draw a generalization. Let L be a real number or $\pm\infty$, and let's now examine the limit $\lim_{x \longrightarrow s}$ where s can take on the values $a, a^{\pm}, \pm\infty$. Note that this gives us 15 different combinations. We can put all of these definitions in the form

$$\forall (\cdot) \exists (\cdot) : (\cdot) \implies (\cdot).$$

If L is a finite value, then the first and last blanks are $\epsilon > 0$ and $|f(x) - L| < \epsilon$, respectively.
If L is $+\infty$, then the first blank is $M > 0$ and the last blank is $f(x) > M$. The $<$ equivalents follow for $L = -\infty$.
If we consider $s = a$, then f is defined on $\mathcal{J}\backslash\{a\}$ and the second and third blanks are $\delta > 0$ and $0 < |x - a| < \delta$.
For $s = a^+$ we consider the interval (a, b) and the second and third blanks are $\delta > 0$ and $a < x < a + \delta$. For $s = a^-$, we consider (c, a) and the second and third blanks are $\delta > 0$ and $a - \delta < x < a$.
For $s = +\infty$ we consider (c, ∞) and the second and third blanks are $\alpha < \infty$ and $\alpha < x$, and a symmetric argument follows for $s = -\infty$.

Theorem: *Let f be a function defined on $\mathcal{J}\backslash\{a\}$ for some open interval $\mathcal{J}$ containing a. Then $\lim_{x \longrightarrow a} f(x)$ exists iff $\lim_{x \longrightarrow a^+} f(x)$ and $\lim_{x \longrightarrow a^-} f(x)$ both exist and are equal.*

Power Series

Given some sequence $(a_n)_{n=0}^{\infty}$ of real numbers,

$$\sum_{n=0}^{\infty} a_n x^n$$

is a *power series.* The power series is a function of f so long it converges for some (or all) x. It is trivial to observe that this converges for $x = 0$. Whether or not it converges for any other x is dependent on our choice of a_n. Given any sequence (a_n), one of the following must be true:

a. The power series converges for all $x \in \mathbb{R}$

b. The power series only converges for $x = 0$

c. The power series converges for all x in some bounded interval (centered at 0). The interval may be open, half-open, or closed.

All these assertions follow from the following theorem:

Theorem: *For the power series* $\sum a_n x^n$*, let*

$$\beta = \limsup |a_n|^{1/n}, R = \frac{1}{\beta}.$$

Then the power series ***converges*** *when* $|x| < R$ *and* ***diverges*** *when* $|x| > R$. We call the number R the *radius of converges* for the power series. If $R = 0$ then the power series cannot converge, and if $R = \infty$ then the power series cannot diverge.

One goal for us to understand *exactly* what the behavior of

$$f(x) = \sum_{k=0}^{\infty} a_k x^k, |x| < R$$

is. Is f continuous? Is it differentiable? If it is, can we differentiate it as

$$f'(x) = \sum_{k=0}^{\infty} k a_k x^{k-1}?$$

How do we know whether f is continuous? Given its partial sums $f_n(x) = \sum_{k=0}^{n} a_k x^k$ (which are continuous as they are polynomials), we may be inclined to state that f is continuous on (a, b) if $\lim_{n\to\infty} f_n(x) = f(x)$. However, this result is false. To see why, examine $f(x) = 0$ for $x \neq 0$ and $f(0) = 1$, and have $f_n(x) = (1 - |x|)^n$. However, power series *do* converges to continuous functions. This is because

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_k x^k \textit{ converges uniformly to } \sum_{k=0}^{\infty} a_k x^k$$

on $[-R_1, R_1]$ where $R_1 < R$.

Uniform Convergence

Definition: Let (f_n) be a sequence of real-valued functions defined on $\mathcal{S} \subseteq \mathbb{R}$. (f_n) *converges pointwise* to f on $\mathcal{S}$ if

$$\lim_{n \to \infty} f_n(x) = f(x) \forall x \in \mathcal{S}$$

Definition: Let (f_n) be a sequence of real-valued functions defined on $\mathcal{S} \subseteq \mathbb{R}$. (f_n) *converges uniformly* to f defined on $\mathcal{S}$ if

$$\forall\, \epsilon > 0 \,\exists\, N : |f_n(x) - f(x)| < \epsilon \,\forall x \in \mathcal{S},\ \forall n > N$$

Theorem: The uniform limit of continuous functions is continuous. Let (f_n) be a sequence of functions on set $\mathcal{S} \subseteq \mathbb{R}$. Suppose $\lim f_n = f$ uniformly on $\mathcal{S}$, and suppose $\mathcal{S} = \text{dom} f$. If each f_n is continuous at $x_0 \in \mathcal{S}$, then f is continuous at x_0. This means that the limiting function f must be continuous if f_n converges uniformly to f.

We can restate the definition of uniform convergence as follows: *A sequence of functions (f_n) converges uniformly to f if and only if*

$$\lim \sup\{|f(x) - f_n(x)| : x \in \mathcal{S}\} = 0$$

Uniform Convergence, contd.

Definition: A sequence (f_n) of functions defined on a set $\mathcal{S} \subseteq R$ is *uniformly Cauchy* if

$$\forall \epsilon > 0 \exists N : |f_n(x) - f_m(x)| < \epsilon \forall x \in \mathcal{S} \forall m, n > N$$

Theorem: *Let (f_n) be a sequence of functions defined and uniformly Cauchy on $\mathcal{S} \subseteq \mathbb{R}$. Then there exists some f on $\mathcal{S}$ such that* $\lim f_n = f$ *uniformly on $\mathcal{S}$.*

The above theorem is especially useful for "series of functions." Recall that

$$\sum_{k=1}^{\infty} a_k$$

only has meaning when

$$\lim \sum_{k=1}^{n} a_k$$

has meaning, i.e. is $\pm\infty$ or a real number. Likewise, a series of functions defined as

$$\sum_{k=1}^{\infty} g_k(x)$$

only has meaning when the sequence of partial sums converges or diverges pointwise, i.e.

$$\lim \sum_{k=1}^{n} g_k(x)$$

exists. If the sequence of partial sums converges uniformly on $\mathcal{S}$, the series is uniformly convergent on $\mathcal{S}$.

Theorem: *Consider the series $\sum_{k=0}^{\infty} g_k$ of functions on $\mathcal{S}$. Suppose g_k is continuous on $\mathcal{S}$ and the series is uniformly convergent on $\mathcal{S}$. Then the series $\sum_{k=1}^{\infty} g_k$ is a continuous function on $\mathcal{S}$.*

Just as we had the Cauchy Criterion for a series $\sum a_k$, we have the Cauchy Criterion defined for a series of functions as well.

$$\forall\, \epsilon > 0\ \exists\, N : n \geq m > N \implies |\sum_{k=1}^{n} g_k(x)| < \epsilon\ \forall x \in \mathcal{S}$$

Theorem: If a series of functions satisfies the Cauchy criterion uniformly on $\mathcal{S}$, then the series converges uniformly on $\mathcal{S}$.

Weierstrass M-test: *Let (M_k) be a sequence of nonnegative real numbers where $\sum M_k < \infty$. If $|g_k(x)| \leq M_k\ \forall x \in \mathcal{S}$, then $\sum g_k$ converges uniformly on $\mathcal{S}$.*

Differentiation and Integration of Power Series

Theorem: *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$. If $0 < R_1 < R$, then the power series converges uniformly on $[-R_1, R_1]$ to a continuous function.*

Corrolary: *The power seris $\sum a_n x^n$ converges to a continuous function on $(-R_1, R_1)$.*

Basic Properties of the Derivative

Definition: Let f be a real valued function defined on an open interval containing a point a. Then f is differentiable at a (equivalently, f has a derivative at a) if the limit

$$\lim_{x \longrightarrow a} \frac{f(x) - f(a)}{x - a}$$

exists and is finite. We express this limit as $f'(a)$. $\text{dom} f'$ is the subset of the domain of f where f is differentiable, meaning that $\text{dom} f' \subseteq \text{dom} f$.

Theorem: *If f is differentiable at a, then f is continuous at a.*

Proof: We are given $f'(a) = \lim_{x \longrightarrow a} \frac{f(x)-f(a)}{x-a}$. We can rewrite $f(x) = (x - a)\frac{f(x)-f(a)}{x-a} + f(a)$. Since $\lim x - a = 0$, this means $\lim f(x) = 0 + f(a)$ meaning $\lim f(x) = f(a)$ and therefore $f(x)$ is continuous at a. ■

The following are a series of general statements about the properties of derivatives, familiar from elementary calculus courses.

Theorem: *Let f, g be functions differentiable at a. Each of the functions cf, $f + g$, fg, and f/g is differentiable at a except f/g in the case where $g(a) = 0$.*

i) $(cf)'(a) = cf'(a)$
ii) $(f + g)'(a) = f'(a) + g'(a)$
iii) (product rule) $(fg)'(a) = f(a)g'(a) + f'(a)g(a)$
iv) (quotient rule) $(f/g)'(a) = [g(a)f'(a) - g'(a)f(a)]/g^2(a)$

Theorem [Chain Rule]: *If f is differentiable at a and g is differentiable at $f(a)$, then the composite $g \circ f$ is differentiable at a and $(g \circ f)(a) = f'(a)g'(f(a))$*

Proof: Assume that f is defined on an open interval $\mathcal{J}$ containing a and that g is defined on an open interval $\mathcal{I}$ containing $f(a)$. We can then assume that $g \circ f$ is defined on $\mathcal{J}$.

Consider a sequence (x_n) defined on $\mathcal{J} \backslash \{a\}$ with $\lim x_n = a$. $\forall n$, let $y_n = f(x_n)$. f is continuous at $x = a$, so $\lim y_n = f(a)$. For each $f(x_n) \neq f(a)$,

$$\frac{(g \circ f)(x_n) - (g \circ f)(a)}{x_n - a} = \frac{g(y_n) - g(f(a))}{y_n - f(a)} \cdot \frac{f(x_n) - f(a)}{x_n - a}$$

Case 1: Suppose $f(x) \neq f(a)$ for x near a. Then $y_n \neq f(a)$ as $n \to \infty$. Then taking the limit of the above yields $(g \circ f)'(a) = g'(f(a))f'(a)$.
Case 2: Suppose $f(x) = f(a)$ for x near a. Then by the Bolzano-Weierstrass theorem there is a sequence $(z_n) \in \mathcal{J} \backslash \{a\}$ such that $\lim z_n = a$ and $f(z_n) = f(a)$. Then $f'(a) = 0$ and then we can see that $(g \circ f)'(a) = 0$. ■

The Mean Value Theorem

To find the maximum and minimum of a continuous function f on the closed interval $[a, b]$, it suffices to evaluate the function at $\{x : f'(x) = 0\}$, points where f is not differentiable, and the endpoints a and b.

Theorem: *If f is defined on an open interval containing x_0, and if f is differentiable at x_0, then $f'(x_0) = 0$.*

Rolle's Theorem: *Let f be a continuous function on $[a, b]$ that is differentiable on (a, b) and satisfies $f(a) = f(b)$. Then there exists at least one $x \in (a, b)$ such that $f'(x) = 0$.*

Mean Value Theorem: *Let f be a continuous function on $[a, b]$ that is differentiable on (a, b). Then there exists at least one $x \in [a, b]$ such that*

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

Corrolary: *Let f be a differentiable function on (a, b) such that $f'(x) = 0$ for all $x \in (a, b)$. Then f is a constant function on (a, b).*

Corrolary: *Let f and g be differentiable functions on (a, b) such that $f' = g'$ on (a, b). Then there exists a constant c such that $f(x) = g(x) + c$ for all $x \in (a, b)$.*
This last corrolary is especially important because it allows us to guarantee that all the antiderivatives (also known as indefinite integrals) for a function differ only by a constant. In other words,

$$\int f'(x) = f(x) + c$$

This also implies that all functions of the form $f(x)+c$ have the same derivative, $f'(x)$.

Definition: Let f be a real-valued function defined on an interval $\mathcal{I}$. We say f is:
strictly increasing on $\mathcal{I}$ if $x_1, x_2 \in \mathcal{I}$, and $x_1 < x_2$ imply $f(x_1) < f(x_2)$
strictly decreasing on $\mathcal{I}$ if $x_1, x_2 \in \mathcal{I}$, and $x_1 < x_2$ imply $f(x_1) > f(x_2)$
increasing on $\mathcal{I}$ if $x_1, x_2 \in \mathcal{I}$, and $x_1 < x_2$ imply $f(x_1) \leq f(x_2)$
decreasing on $\mathcal{I}$ if $x_1, x_2 \in \mathcal{I}$, and $x_1 < x_2$ imply $f(x_1) \geq f(x_2)$

Corrolary: *Let f be a differentiable function on an interval (a, b). Then*
f is strictly increasing if $f'(x) > 0$ on (a, b)
f is strictly decreasing if $f'(x) < 0$ on (a, b)
f is increasing if $f'(x) \geq 0$ on (a, b)
f is decreasing if $f'(x) \leq 0$ on (a, b)

Intermediate Value Theorem for Derivatives: *Let f be a differentiable function on (a, b). If $a < x_1 < x_2 < b$ and if c lies between $f'(x_1)$ and $f'(x_2)$, there exists at least one $x \in (x_1, x_2)$ such that $f'(x) = c$.*

Theorem: *Let f be a one-to-one continuous function on an open interval $\mathcal{I}$, and let $\mathcal{J} = f(\mathcal{I})$. If f is differentiable at $x_0 \in \mathcal{I}$ and if $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and*

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

L'Hospital's Rule

We are frequently faced with taking limits of the quotient of functions, i.e. of the form

$$\lim_{x \longrightarrow s} \frac{f(x)}{g(x)}$$

If the limits of the individual functions exist, the resultant limit is simply

$$\frac{\lim_{x \longrightarrow s} f(x)}{\lim_{x \longrightarrow s} g(x)}$$

However, we begin to run into problems when we are faced with *indeterminate forms*, i.e. when the quotient of the limits is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

The Generalized Mean Value Theorem: *Let f and g be continuous functions on $[a, b]$ that are differentiable on (a, b). Then there exists $\geq 1 x \in (a, b)$ such that*

$$f'(x)[g(b) - g(a)] = g'(x)[f(b) - f(a)]$$

When $g(x) = x$, this generalizes to the standard Mean Value Theorem.

L'Hospital's Rule: *Let s signify a, $-a$ $+\infty$, $-\infty$ where $a \in \mathbb{R}$, and suppose f and g are differentiable functions for which the following limit exists:*

$$\lim_{x \longrightarrow s} \frac{f'(x)}{g'(x)} = L.$$

If

$$\lim_{x \longrightarrow s} f(x) = \lim_{x \longrightarrow s} g(x) = 0$$

or if

$$\lim_{x \longrightarrow s} |g(x)| = +\infty,$$

then

$$\lim_{x \longrightarrow s} \frac{f(x)}{g(x)} = L.$$

Taylor's Theorem

Consider a power series with radius of convergence $R > 0$, where R may be $+\infty$.

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

f is differentiable on the interval $|x| < R$ and

$$f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}.$$

However, we can continue differentiating and get that

$$f''(x) = \sum_{2}^{\infty} k(k-1) a_k x^{k-2}$$

Inductively applying this tells us that

$$f^{(n)}(x) = \sum_{k=n}^{\infty} k(k-1)\dots(k-n+1) a_n x^{k-n}$$

Additionally note that

$$f^{(n)}(0) = n(n-1)\dots(n-n+1)a_n = n! a_n.$$

We can extrapolate this to the original function to see that

$$f(x) = \sum_{k=1}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

Definition: Let f be a function defined on some open interval contaiing c. If f possesses derivatives of all orders at c, then the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

is called the *Taylor series for f about c*. For $n \geq 1$, the *remainder* $R_n(x)$ is defined as

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

The remainder is useful because our Taylor Series only converges to our function if $\lim_{n\to\infty} R_n(x) = 0$.

Taylor's Theorem: *Let f be defined on (a, b) where $a < c < b$. Here we* *alllow $a = -\infty$ or $b = \infty$. Suppose the nth derivative $f^{(n)}$ exists on (a, b). Then for each $x \neq c$ on (a, b) there is some y between c and x such that*

$$R_n(x) = \frac{f^{(n)}(y)}{n!}(x-c)^n.$$

Corrolary: *Let f be defined on (a, b) where $a < c < b$. If all the derivaties of $f^{(n)}$ exist on (a, b) and are bounded by a single constant C, then*

$$\lim_{n \longrightarrow \infty} R_n(x) = 0 \text{ for all } x \in (a, b).$$

We now look for a form of Taylor's Theorem which offers the remainder as an integral.

Taylor's Theorem, Integral Form: *Let f be defined on (a, b) where $a < c < b$. Suppose the nth derivative exists and is continuons on (a, b). Then for $x \in (a, b)$*

$$R_n(x) = \int_c^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)dt.$$

Cauchy Remainder: *If f is as in the previous theorem then for each $x \in (a, b)$ different from c there is some y between c and x such that*

$$R_n(x) = (x-c) \cdot \frac{(x-y)^{n-1}}{(n-1)!} f^{(n)}(y).$$

This is known as the *Cauchy form* of the remainder.

Recall that for some nonnegative n, the binomial theorem tells us that

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

For $a = x$ and $b = 1$, we can derive the following.

Binomial Series Theorem: *If $\alpha \in \mathbb{R}$ and $|x| < 1$, then*

$$(1+x)^\alpha = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} x^k.$$

Newton's Method

Newton's method for approximating $f(x) = 0$is to begin with some reasonable guess, x_0, and then compute

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

Secant Method

The secant method for approximating $f(x) = 0$ is similar; we begin with two reasonable guesses x_0 and x_1 and compute

$$x_n = x_{n-1} - \frac{f(x_{n-1})(x_{n-2} - x_{n-1})}{f(x_{n-2}) - f(x_{n-1})}$$

We now need a lemma that assures us that Newton's method will give a good approximation for $f(x) = 0$.

Lemma: *Let (a_n) be a sequence of nonnegative numbers, and let C and δ be positive numbers satisfying $C\delta < 1$.If $a_0 \le \delta$ and $a_n \le C a_{n-1}^2$ for $n \ge 1$, then*

$$a_n \le (C\delta)^{2^n - 1} a_0 \ \forall\, n \ge 0.$$

If $\max\{a_0, a_1\} \le \delta$ and $a_n \le C \max\{a_{n-1}, a_{n-2}\}^2$ for $n \ge 2$, then

$$\max\{a_{2n}, a_{2n+1}\} \le (C\delta)^{2^n - 1} \max\{a_0, a_1\} \ \forall\, n \ge 0.$$

Theorem: *Consider a function f having a zero $\bar{x}$ on an interval $\mathcal{J} = (c, d)$, and assume f'' exists on $\mathcal{J}$. Assume $|f''|$ is bounded above on $\mathcal{J}$ and $|f'|$ is bounded away from 0 or $\mathcal{J}$. Choose $\delta_0 > 0$ so that $\mathcal{I} = [c + \delta_0, d - \delta_0] \subset \mathcal{J}$ is a nondegenerate interval containing $\bar{x}$ and so that $[c + \delta_0, d - \delta_0] \subseteq \mathcal{J}$. Let*

$$C = \frac{\sup\{|f''(x)| : x \in \mathcal{J}\}}{2 \inf\{|f'(x)| : x \in \mathcal{J}\}},$$

and select $\delta > 0$ so that $2\delta \le \delta_0$ and $C\delta < 1$. Let $m = \inf\{|f'(x)| : x \in \mathcal{J}\}$. Consider any $x_0 \in \mathcal{I}$ satisfying $|f(x)| < m\delta$. Then the sequence of iterates given by Newton's method,

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})},$$

is a well-defined sequence and converges to $\bar{x}$. Also,

$$|x_n - \bar{x}| \le C|x_{n-1} - \bar{x}|^2 \text{ and}$$

$$|x_n - \bar{x}| \le (C\delta)^{2^n - 1} |x_0 - \bar{x}|.$$

We call (x_n) *quadratically convergent.* A similar theorem follows for the secant method:

Theorem: *Assume the notation and hypotheses of the previous theorem. Here, we let*

$$C = \frac{3 \sup\{|f''(x)| : x \in \mathcal{J}\}}{2 \inf\{|f'(x)| : x \in \mathcal{J}\}}$$

and consider distince $x_0, x_1 \in \mathcal{I}$ with

$$\max\{|f(x_0)|, |f(x_1)|\} < m\delta.$$

Choose δ as in the previous theorem. The sequence of iterates yielded by the secant method,

$$x_n = x_{n-1} - \frac{f(x_{n-1})(x_{n-2} - x_{n-1})}{f(x_{n-2}) - f(x_{n-1})},$$

is well defined and converges to $\bar{x}$. Also

$$|x_n - x\bar{x}| \leq C \cdot \max\{|x_{n-1} - \bar{x}|, |x_{n-2} - \bar{x}|\}^2 \text{ and}$$

$$\max\{|x_{2n} - \bar{x}|, |x_{2n+1} - \bar{x}|\} \leq (C\delta)^{2^n - 1} \max\{|x_0 - \bar{x}|, |x_1 - \bar{x}|\}.$$

The Riemann Integral

Here we begin to develop the Riemann integral, the common integral in standard calculus courses.

The Darboux Integral: Let f be a bounded function on $[a, b]$. For $\mathcal{S} \subseteq [a, b]$ we adopt the notation $M(f, \mathcal{S}) = \sup\{f(x) : x \in \mathcal{S}\}$ and $m(f, \mathcal{S}) = \inf\{f(x) : x \in \mathcal{S}\}$. A *partition of* $[a, b]$ is any finite ordered subset $\mathcal{P}$ of the form

$$\mathcal{P} = \{a = t_0 < t_1 < \ldots < t_n = b\}.$$

The *upper Darboux sum* $U(f, \mathcal{P})$ of f with respect to $\mathcal{P}$ is the sum

$$U(f, \mathcal{P}) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1})$$

The *lower Darboux sum* $L(f, \mathcal{P})$ is

$$L(f, \mathcal{P}) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1})$$

Note that

$$U(f, \mathcal{P}) \leq \sum_{k=1}^{n} M(f, [a, b]) \cdot (t_k - t_{k-1}) = M(f, [a, b]) \cdot (b - a)$$

Likewise,

$$L(f, \mathcal{P}) \geq m(f, [a, b]) \cdot (b - a).$$

Then

$$m(f, \mathcal{P}) \cdot (b - a) \leq L(f, \mathcal{P}) \leq U(f, \mathcal{P}) \leq M(f, \mathcal{P}) \cdot (b - a) \quad \textbf{(1)}$$

The *upper Darboux integral* is defined as

$$\overline{\int_a^b} f = U(f) = \inf\{U(f, \mathcal{P}) : \mathcal{P} \text{ is a partition of } [a, b]\}$$

and the *lower Darboux integral* is

$$\underline{\int_a^b} f = L(f) = \sup\{L(f, \mathcal{P}) : \mathcal{P} \text{ is a partition of } [a, b]\}$$

We say that f is integrable on $[a, b]$ if $L(f) = U(f)$. In this case we can define the *Darboux integral* as

$$\int_a^b f = \int_a^b f(x)dx = \underline{\int_a^b} f = \overline{\int_a^b} f$$

We will show later that the definition of the Darboux integral is equivalent to the definition of the **Riemann integral** and we will use the latter term to refer to the above formulation.

We can geometrically interpret $\int_a^b f$ as the area under a region of the graph f. Each lower Darboux integral is the area of a union of rectangles underneath the function, and the upper Darboux integral is the area of a union of rectangles containing the function. When these two values are the same, we must have the area underneath the function by the squeeze theorem.

Lemma: *Let f be a bounded function on $[a, b]$. If $\mathcal{P}$ and $\mathcal{Q}$ are partitions of $[a, b]$ and $\mathcal{P} \subseteq \mathcal{Q}$, then*

$$L(f, \mathcal{P}) \leq L(f, \mathcal{Q}) \leq U(f, \mathcal{Q}) \leq U(f, \mathcal{P})$$

Lemma: *If f is a bounded function on $[a, b]$, and if $\mathcal{P}$ and $\mathcal{Q}$ are partitions of $[a, b]$, then $L(f, \mathcal{P}) \leq U(f, \mathcal{Q})$.*

Theorem: *If f is a bounded function on $[a, b]$, then $L(f) \leq U(f)$.*

Thorem: *A bounded function f on $[a, b]$ is integrable if and only if for each $\epsilon > 0$ there exists a partition $\mathcal{P}$ of $[a, b]$ such that*

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) \leq \epsilon.$$

Definition: The *mesh* of a partition $\mathcal{P}$ is the maximum length of the subintervals comprisin $\mathcal{P}$. Thus if

$$\mathcal{P} = \{a = t_0 < t_1 < \ldots < t_n = b\}$$

then

$$\text{mesh}(\mathcal{P}) = \max\{t_k - t_{k-1} : k = 1, 2, ..., n\}.$$

Theorem: Cauchy Criterion for Integrability: *A bounded function f on $[a, b]$ is integrable if and only if for each $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$\textit{mesh}(\mathcal{P}) < \delta \implies U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon$$

for all partitions $\mathcal{P}$ of $[a, b]$.

We now offer the Riemann definition of integrability.

Definition: Let f be a bounded function on $[a, b]$, and let $\mathcal{P} = \{a = t_0 < t_1 < \ldots < t_n = b\}$ be a partition of $[a, b]$. A *Riemann sum* of f associated with the partition $\mathcal{P}$ is a sum of the form

$$\sum_{k=1}^{n} f(x_k)(t_k - t_{k-1})$$

where $x_k \in [t_{k-1}, t_k]$ for $k = 1, 2, ..., n$. The choice of x_k doesn't matter, so there are an infinite number of Riemann sums associated with a single function.

A function is *Riemann integrable* on the interval $[a, b]$ if

$$\exists r : \forall \epsilon > 0 \; \exists \; \delta > 0 : \forall \mathcal{S}, \text{mesh}(\mathcal{P}) < \delta \implies |\mathcal{S} - r| < \epsilon$$

where $\mathcal{S}$ is the set of Riemann sums of f with partition $\mathcal{P}$.

Theorem: *A function f is Riemann integrable if and only if it is Darboux integrable, in which case the values of the integrals coincide.*

Corrolary: *Let f be a bounded Riemann integrable function on $[a, b]$. Suppose $(\mathcal{S}_n)$ is a sequence of Riemann sums, with partitions $\mathcal{P}_n$, where $\lim_n$ mesh$(\mathcal{P}_n) = 0$. Then $(\mathcal{S}_n)$ converges to $\int_a^b f$.*

Properties of the Riemann Integral

Recall that a function is *monotonic* on an interval if it is entirely increasing or decreasing on that interval.
Theorem: *Every monotonic function f on $[a,b]$ is integrable.*

Theorem: *Every continuous function f on $[a,b]$ is integrable.*

Theorem: *Let f and g be integrable functions on $[a,b]$, and let c be a real number. Then*

i) cf is integrable and $\int_a^b cf = c\int_a^b f$;

ii) $f+g$ is integrable and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

Theorem: *If f and g are integrable on $[a,b]$ and if $f(x) \le g(x)$ for $x \in [a,b]$, then $\int_a^b f \le \int_a^b g$.*

Theorem: *If g is a continuous nonnegative funtion on $[a,b]$ and if $\int_a^b g = 0$, then g is identically 0 on $[a,b]$.*

Theorem: *If f is integrable on $[a,b]$, then $|f|$ is integrable on $[a,b]$ and*

$$\left|\int_a^b f\right| \le \int_a^b |f|$$

Theorem: *Let f be a function defined on $[a,b]$. If $a < c < b$ and f is integrable on $[a,c]$ and $[c,b]$, then f is integrable on $[a,b]$ and*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Definition: A function f on $[a,b]$ is *piecewise monotonic* if there is a partition

$$\mathcal{P} = \{a = t_0 < ... < t_n = b\}$$

of $[a,b]$ such that f is monotonic on each interval (t_{k-1}, t_k). The function is *piecewise continuous* if there is a partition $\mathcal{P}$ of $[a,b]$ such that f is uniformly continuous on each interval (t_{k-1}, t_k).

Theorem: *If f is piecewise continuous or piecewise monotonic on $[a,b]$, then f is integrable on $[a,b]$.*

Intermediate Value Theorem of Integrals: If f is a continuous function on $[a,b]$, then for at least one $x \in (a,b)$ we have:

$$f(x) = \frac{1}{b-a}\int_a^b f$$

Dominated Convergence Theorem: Suppose (f_n) is a sequence of integrable functions on $[a,b]$ and $f_n \to f$ pointwise where f is an integrable function on $[a, b]$. If there exists some $M > 0$ such that $|f_n(x)| \leq M$ for all n and all $x \in [a, b]$, then

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx.$$

Monotone Convergence Theorem Suppose (f_n) is a sequence of integrable funcrtions on $[a, b]$ such that $f_1(x) \leq f_2(x) \leq ... \ \forall x \in [a, b]$. Suppose also that $f_n \to f$ pointwise where f is integrable on $[a, b]$. Then

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx.$$

Fundamental Theorem of Calculus

There are two versions of the FTC. Each says that the operations of *integration* and *differentiation* are basically inverse operations. The first version states that the integral of the derivative of a function is the function; the second states that the derivative of the integral of a function is the function.

We say that a function h on (a, b) is *integrable* on $[a, b]$ if every extension of h to $[a, b]$ is integrable. Note that the value of $\int_a^b h$ does not depend on $\tilde{h}(a)$ or $\tilde{h}(b)$.

Fundamental Theorem of Calculus I: If g is a continuous function on $[a, b]$ that is differentiable on (a, b), and if g' is integrable on $[a, b]$, then

$$\int_a^b g' = g(b) - g(a).$$

Proof: Let $\epsilon > 0$. We know that a bounded function f on (a, b) is integrable if and only if for each $\epsilon > 0$ there exists a partition $\mathcal{P}$ of $[a, b]$ such that

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon,$$

where U and L are the upper and lower Darboux sums, respectively. Since g' is integrable, we know that

$$U(g', \mathcal{P}) - L(g', \mathcal{P}) < \epsilon$$

for a partition $\mathcal{P} = \{a = t_0 < ... < t_n = b\}$. We can use the Mean Value Theorem and find $x_k \in (t_k, t_{k-1})$ for which

$$(t_k - t_{k-1})g'(x_k) = g(t_k) - g(t_{k-1})$$

Then we have that

$$g(b) - g(a) = \sum_{k=1}^{n} g(t_k) - g(t_{k-1}) = \sum_{k=1}^{n} g'(x_k)(t_k - t_{k-1}).$$

We then have that

$$L(g', \mathcal{P}) \leq g(b) - g(a) \leq U(g', \mathcal{P});$$

this comes from our previous formal definition of the integral. Since

$$L(g', \mathcal{P}) \leq \int_a^b g' \leq U(g', \mathcal{P}),$$

we have the solution that

$$\left| \int_a^b g' - [g(b) - g(a)] \right| < \epsilon.$$

Theorem: Integration by Parts: *If u and v are continuous functions on $[a, b]$ that are differentiable on (a, b), and if u' and v' are integrable on $[a, b]$, then*

$$\int_a^b u(x)v'(x)dx + \int_a^b u'(x)v(x)dx = u(b)v(b) - u(a)v(a)$$

■

Fundamental Theorem of Calculus II: Let f be an integrable function on $[a, b]$. For x in $[a, b]$, let

$$F(x) = \int_a^x f(t)dt.$$

Then F is continuous on $[a, b]$. If f is continuous at $x_0 \in (a, b)$, then F is differentiable at x_0 and

$$F'(x_0) = f(x_0).$$

Theorem: Change of Variable: *Let u be a differentiable function on an open interval $\mathcal{J}$ such that u' is continuous, and let $\mathcal{I}$ be an open interval such that $u(x) \in \mathcal{I}$ for all $x \in \mathcal{J}$. If f is continuous on $\mathcal{I}$, then $f \circ u$ is continuous on $\mathcal{J}$ and*

$$\int_a^b f \circ u(x)u'(x)dx = \int_{u(a)}^{u(b)} f(u)du.$$