MATH140 Notes

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Introduction

This course is an elementary introduction to differential geometry. Differential geometry is the study of geometry using differential and integral calculus along with linear and multilinear algebra. These notes are based on lectures by Dr. John Lott and *Elements of Differential Geometry* by Richard Millman and George Parker. This is an introductory class at an undergraduate level, but it is necessary to have a strong understanding of elementary real analysis, linear algebra, and multivariable calculus before attempting this course. Additionally, to reduce complexity, the majority of this course remains in $\mathbb{R}^3$ without straying to generalizations, and avoids unnecessarily complicated machinery such as cohomology and differential forms.

Contents

1 Preliminaries

Differential geometry combines linear algebra and calculus. Here is a brief summary of the key concepts of these subjects.

1.1 Vector Spaces

A **real vector space** is a set (typically denoted V) whose elements are called **vectors** along with the binary operations of **addition** and **scalar multiplication**. Vector spaces must satisfy the following 8 **vector space axioms** for all vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in V$ and $r, s \in \mathbb{R}$.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity of addition);
2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (associativity of addition);
3. There is a (unique) $\mathbf{0}$ such that $\mathbf{0} + \mathbf{u} = \mathbf{u}$ (zero property);
4. $(rs)\mathbf{u} = r(s\mathbf{u})$ (associativity of scalar multiplication);
5. $(r + s)\mathbf{u} = r\mathbf{u} + s\mathbf{u}$ (distributivity of scalar multiplication);
6. $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$ (distributivity of vector addition);
7. $0\mathbf{u} = \mathbf{0}$ (zero vector);
8. $1\mathbf{u} = \mathbf{u}$ (identity).

Three-dimensional space ($\mathbb{R}^3$) and the set of polynomials with real coefficients ($\mathbb{R}[x]$) are classic examples of real vector spaces.

Basis

A set of vectors $\{\mathbf{v}_i : i \in I\} \subset V$ is **linearly independent** if every finite linear combination $\sum a_i \mathbf{v}_i$ is only zero if every a_i is zero. A subset $S \subset V$ **spans** V if, for each $\mathbf{v} \in V$, there are vectors $\{v_i : i \in I\}$ and coefficients $\{a_i : i \in I\}$ such that $\sum a_i \mathbf{v}_i = \mathbf{v}$.

A **basis** of a vector space is a linearly independent spanning set. The number of elements in a basis is the **dimension** of V.

Inner Product

The **inner product** on a vector space V is a function $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ such that

1. $\langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{u}\rangle$;
2. $\langle \mathbf{u}, r\mathbf{v} + s\mathbf{w}\rangle = r\langle \mathbf{u}, \mathbf{v}\rangle + s\langle \mathbf{u}, \mathbf{w}\rangle$;
3. $\langle \mathbf{u}, \mathbf{u}\rangle \geq 0$ with equality iff $\mathbf{u} = 0$.

In $\mathbb{R}^3$ we use the ordinary dot product; in $\mathbb{R}[x]$ we typically use $\langle p(x), q(x)\rangle = \int_{-1}^{1} p(x)q(x)dx$. The **length** of $\mathbf{v}$ is $|\mathbf{v}| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}$.

Cauchy-Schwarz Inequality

If $\mathbf{u}, \mathbf{v} \in V$, then $|\langle \mathbf{u}, \mathbf{v}\rangle| \leq |\mathbf{u}||\mathbf{v}|$, where equality exists only if $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent.

This tells us the **angle** between two vectors, θ, can be described by $\langle \mathbf{u}, \mathbf{v}\rangle = |\mathbf{u}||\mathbf{v}|\cos\theta$. We call two vectors $\mathbf{u}$ and $\mathbf{v}$ **orthogonal** if $\langle \mathbf{u}, \mathbf{v}\rangle = 0$; a basis composed entirely of mutually orthogonal vectors is **orthonormal**.

1.2 Linear Transformations and Eigenvectors

A **linear transformation** is a function $T : V \to W$ of vector spaces such that A **linear transformation** is a function $T : V \to W$ of vector spaces such that $T(a\mathbf{v} + b\mathbf{w}) = aT(\mathbf{v}) + bT(\mathbf{w})$. An **isomorphism** is a bijective linear transformation.

Suppose $T : V \to V$ is a linear transformation and V has two bases $\{\mathbf{u}_i\}_{i=1}^n$ and $\{\mathbf{v}_\alpha\}_{\alpha=1}^n$, related by $\mathbf{u}_i = \sum a_{\alpha_i}\mathbf{v}_\alpha$. Then if (T_{ij}) is the transformation with respect to the basis $\mathbf{u}_i$ and $(T_{\alpha\beta})$ is the transformation with respect to the basis $\mathbf{v}_\alpha$, then Suppose $T : V \to V$ is a linear transformation and V has two bases $\{\mathbf{u}_i\}_{i=1}^n$ and $\{\mathbf{v}_\alpha\}_{\alpha=1}^n$, related by $\mathbf{u}_i = \sum a_{\alpha_i}\mathbf{v}_\alpha$. Then if (T_{ij}) is the transformation with respect to the basis $\mathbf{u}_i$ and $(\overline{T}_{\alpha\beta})$ is the transformation with respect to the basis $\mathbf{v}_\alpha$, then the transformations are related by

$$(T_{ij}) = (a_{\alpha i})^{-1}(\overline{T}_{\alpha\beta})(a_{\beta j}).$$

Eigenvalues

Let $T : V \to V$ be a linear transformation. A real number λ is an **eigenvalue** of T if there is a nonzero vector $\mathbf{v}$ (the **eigenvector**) such that $T\mathbf{v} = \lambda\mathbf{v}$. The eigenvalues of (T_{ij}) are the real solutions of the polynomial $\det(T_{ij} - x\delta_{ij}) = 0$ (here δ is the Kroneker delta). The number of eigenvalues is at most the dimension of V.

1.3 Orientation and Cross Products

Let $\{\mathbf{u}_i\}_{i=1}^n$ and $\{\mathbf{v}_i\}_{i=1}^n$ be two **ordered** bases (meaning we preserve the ordering of the sets). Define a matrix (a_{ij}) by $\mathbf{v}_j = \sum a_{ij}\mathbf{u}_i$. The ordered bases have the **same orientation** if $\det(a_{ij}) > 0$, and have the **opposite orientation** if $\det(a_{ij}) < 0$. From this point forward we call the orientation of the **standard basis** in $\mathbb{R}^3$ $(\{(1,0,0),(0,1,0),(0,0,1)\})$ **right handed**; we shorthand it as $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

Cross Product

If $\mathbf{u} = \sum a_i\mathbf{e}_i$ and $\mathbf{v} = \sum b_i\mathbf{e}_i$ in $\mathbb{R}^3$, then the **cross product** $\mathbf{u} \times \mathbf{v}$ is

$$\mathbf{u} \times \mathbf{v} = \mathbf{e}_1(a_2b_3 - a_3b_2) + \mathbf{e}_2(a_3b_1 - a_1b_3) + \mathbf{e}_3(a_1b_2 - a_2b_1).$$

The following properties of the cross product hold:

1. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$;
2. $(r\mathbf{u}) \times \mathbf{v} = r(\mathbf{u} \times \mathbf{v})$;
3. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ iff $\mathbf{u}$ and $\mathbf{v}$ are dependent;
4. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$;
5. $\mathbf{u} \times \mathbf{v} \perp \mathbf{u}$ and $\mathbf{u} \times \mathbf{v} \perp \mathbf{v}$;
6. $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$;
7. $\{\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v}\}$ gives a right-handed orientation if $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

The **triple product** of $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ is $[\mathbf{u}, \mathbf{v}, \mathbf{w}] = \langle(\mathbf{u} \times \mathbf{v}), \mathbf{w}\rangle$ and its absolute value is the volume of the parallelopiped spanned by $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

1.4 Lines, Planes, and Spheres

This is a brief review of lines, planes, and spheres, from multivariable calculus. A straight line is defined by a point on the line and a vector parallel to the line. A plane is a point on the plane and a vector perpendicular to the plane. A sphere is defined by its center and its radius.

A **line** through $\mathbf{x}_0 \in \mathbb{R}^3$ and parallel to $\mathbf{v} \neq \mathbf{0}$ has equation $\alpha(t) = \mathbf{x}_0 + t\mathbf{v}$. The line through points $\mathbf{x}_1$ and $\mathbf{x}_2$ has equation $\alpha(t) = \mathbf{x}_1 + t(\mathbf{x}_2 - \mathbf{x}_1)$.

The **plane** through $\mathbf{x}_0$ perpendicular to the vector $\mathbf{n} \neq \mathbf{0}$ has equation $\langle \mathbf{x} - \mathbf{x}_0, \mathbf{n} \rangle = 0$. Since the cross product of linearly independent vectors is perpendicular to both vectors, this implies $\langle \mathbf{x} - \mathbf{x}_0, \mathbf{u} \times \mathbf{v} \rangle = 0$ is the plane through $\mathbf{x}_0$ parallel $\mathbf{u}$ and $\mathbf{v}$.

The **sphere** in $\mathbb{R}^3$ with center $\mathbf{m}$ and radius $r > 0$ has equation $\langle \mathbf{x} - \mathbf{m}, \mathbf{x} - \mathbf{m} \rangle = r^2$.

1.5 Vector Calculus

Let $\mathbf{f} : \mathbb{R} \to V$. Let $\{\mathbf{v}_i\}_{i=1}^n$ be a basis for V, so $\mathbf{f} = \sum f_i(t)\mathbf{v}_i$. If each f_i is integrable or differentiable, we may integrate or differentiate $\mathbf{f}$ elementwise:

$$\frac{d\mathbf{f}}{dt} = \sum \frac{df_i}{dt}\mathbf{v}_i$$

$$\int_a^b \mathbf{f}(t)dt = \sum \left(\int_a^b f_i(t)dt \right) \mathbf{v}_i.$$

We may additionally take partial derivatives and multiple integrals in an analogous manner.

Let $\mathbf{f}$ and $\mathbf{g}$ be vector-valued functions into an inner product space V. Then

$$\frac{f}{dt} \langle \mathbf{f}, \mathbf{g} \rangle = \left\langle \frac{d\mathbf{f}}{dt}, \mathbf{g} \right\rangle + \left\langle \mathbf{f}, \frac{d\mathbf{g}}{dt} \right\rangle.$$

Similarly, analgous to the product rule:

$$\frac{d}{dt}(\mathbf{f} \times \mathbf{g}) = \frac{d\mathbf{f}}{dt} \times \mathbf{g} + \mathbf{f} + \frac{d\mathbf{g}}{dt}.$$

A polynomial is of **class** C^k if if it is k-times differentiable, meaning that all its mixed partial derivatives of order k or less are continuous. Finally, consider the chain rule for vector calculus: if a function $\mathbf{x}$ consists of several variables $\{u_i\}_{i=1}^n$, and each u_i is a function of several variables $\{v_i\}_{i=1}^m$, then

$$\frac{\partial \mathbf{x}}{\partial v_\alpha} = \sum_{i=1}^n \frac{\partial \mathbf{x}}{\partial u_i}\frac{\partial u_i}{\partial v_\alpha}, \quad \alpha = 1, ..., m.$$

2 Local Curve Theory

There are two ways to think of curves; we can think of a curve as a geometric set of points (or **locus**), or as the path of a particle in $\mathbb{R}^3$ (the function of a parameter). The latter is required to apply methods of calculus to the curve, but the former also allows us to examine geometric properties of the curve (tangent field).

2.1 Definitions

We restrict our study to specific curves. If $d\alpha/dt$ is 0 on an interval then the curve is constant, which is geometrically uninteresting. If $d\alpha/dt$ is 0 at a single point, then the curve could have a sharp corner, which is also a case we would like to ignore.

Regular Curves

A **regular curve** in $\mathbb{R}^3$ is a function $\alpha : (a, b) \to \mathbb{R}^3$ which is of class C^k for some $k \geq 1$ and for which $d\alpha/dt \neq \mathbf{0}$ for all $t \in (a, b)$. Note that this describes the *function*, not the **geometry** of the curve; two curves with the same geometry may have different parametrizations.

Using this notion of parametrization, we may define **vector fields** along α; this means that for each t, we define a vector $\mathbf{v}(t)$ originating at $\alpha(t)$.

Velocity

The **velocity vector** of a regular curve $\alpha(t)$ at $t = t_0$ is the derivative $d\alpha/dt$ evaluated at $t = t_0$. The **velocity vector field** is the vector valued function $d\alpha/dt$. The **speed** of $\alpha(t)$ at $t = t_0$ is the magnitude of the velocity vector, $(d\alpha/dt)(t_0)$.

Tangent

The **tangent vector field** to a regular curve $\alpha(t)$ is the vector-valued function $\mathbf{T}(t) = (d\alpha/dt)/|d\alpha/dt|$; this is the velocity vector field with all vectors normalized to unit length. This is a geometric property, i.e. it is independent of the parametrization of α.

The **tangent line** to a regular curve α at the point $t = t_0$ is the straight line

$$l = \{\mathbf{w} \in \mathbb{R}^3 : \mathbf{w} = \alpha(t_0) + \lambda\mathbf{T}(t_0), \lambda \in \mathbb{R}\}.$$

Reparametrization

A **reparametrization** of a curve $\alpha : (a, b) \to \mathbb{R}^3$ is a one-to-one function $g : (a, b) \to (c, d)$ such that both g and its inverse $h : (c, d) \to (a, b)$ are of class C^k for $k \geq 1$.

In particular, we are curious about the new curve $\beta = \alpha \circ g$. If r is the parameter in (c, d), then $d\beta/dr = (d\alpha/dt)(dg/dr)$. From this we may conclude that the composition of a regular curve with a reparametrization is also regular. Any property of a curve that does not change after a reparametrization is known as being **geometrically invariant**. We now show that the tangent vector field is geometrically invariant. In particular, for a regular curve α and its reparametrization β, and $t_0 = g(r_0)$, the tangent field of α, $\mathbf{T}(t_0)$, and the tangent field of β, $\mathbf{S}(r_0)$, satisfy $\mathbf{S} = \pm\mathbf{T}$.

Proof:

$$\begin{aligned}
\mathbf{S} &= \frac{d\beta/dr}{|d\beta/dr|} \\
&= \frac{(d\alpha/dt)(dg/dr)}{|d\alpha/dt||dg/dr|} \\
&= \frac{d\alpha/dt}{|d\alpha/dt|}\frac{dg/dr}{|dg/dr|} \\
&= (\mathbf{T})(\pm 1) \\
&= \pm\mathbf{T}.
\end{aligned}$$

□

2.2 Arc Length

Here we consider some curves on closed intervals. A **regular curve segment** is a function $\alpha : [a,b] \to \mathbb{R}^3$ together with an open interval (c,d) with $c < a < b < d$ and a regular curve $\gamma : (c,d) \to \mathbb{R}^3$ such that $\alpha(t) = \gamma(t)$ for all $t \in [a,b]$. The reason we make this definition is so that we can examine a curve on a closed interval and still be able to find the derivative at the endpoints of α.

Length of a Regular Curve Segment

The **length** of a regular curve segment $\alpha : [a,b] \to \mathbb{R}^3$ is

$$\int_a^b \left| \frac{d\alpha}{dt} \right| dt.$$

If α is reparametrized by g, then the length of α is equal to the length of $\beta = \alpha \circ g$ (showing that length is a geometric property). Additionally, say we have a point $t_0 \in (a,b)$. Then the length of the segment from t_0 to t, $h(t) = \int_{t_0}^t |d\alpha/dt| dt$, is the **signed arc length** from t_0 to t and is a reparametrization, called **parametrization by arc length** (also called the **natural parametrization**).

If $\beta(s)$ $(s = h(t))$ is parametrized by arc length, then its velocity vector field is a unit vector field and is thus equal to its tangent vector field. We then call β a **unit speed curve**.

If $\alpha(t)$ is a regular curve and $s = s(t)$ is its arc length, then

1. $s = s(t) = \int_0^t |d\alpha/dt| dt$;
2. $ds/dt = |d\alpha/dt|$;
3. $d\alpha/dt = (ds/dt)\mathbf{T}$; and
4. $\mathbf{T} = d\alpha/ds$.

While arc length parametrization is useful, it is often difficult to compute. What this discussion *does* do is tell us that, if we only care about the geometry of the curve (and not its parametrization), we can arbitrarily assume that the curve is arc length parametrized.

2.3 Curvature and the Frenet-Serret Apparatus

A curve $\alpha : (a,b) \to \mathbb{R}^3$ is a **unit-speed curve** if $|d\alpha/dt| = 1$. For a unit speed curve $\alpha = \alpha(t)$, let $s = t$ be the arc length. Note that $\alpha' = \alpha'(s) = \mathbf{T}(s)$. We can do this because (in the previous section) we asserted that any regular curve can be reparametrized by arc length.

We now need a way to define how to measure **curvature**, the central principle of differential geometry. Intuitively, curvature should be some measure of how much a curve *bends* – a straight line should have no curvature, and a circle should have the same curvature everywhere.

Curvature

The **curvature** of a unit speed curve $\alpha(s) = \kappa(s) = |\mathbf{T}'(s)|$. $\kappa(s) = 0$ if $\alpha(s)$ is a straight line and $1/r$ if α is a circle with radius r.

If we have a point on a curve, and draw every single 3-D vector whose basepoint is that point, we end up with a 3-dimensional vector space. How do we best describe this vector space? A typical answer would be to take the vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the standard basis of $\mathbb{R}^3$. However this is a basis that reflects the geometry of $\mathbb{R}^3$, not the geometry of the curve. Our method will be as follows: take the one vector we already know (the tangent vector field), find another one (the normal vector field) and use their cross product (the binormal vector field).

Torsion

The **principal normal vector field** to a unit speed curve $\alpha(s)$ is the (unit) vector field $\mathbf{N}(s) = \mathbf{T}'(s)/\kappa(s)$. The **binormal vector field** to $\alpha(s)$ is $\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$. The **torsion** of α is the real valued function

$$\tau(s) = -\left\langle \mathbf{B}'(s), \mathbf{N}(s) \right\rangle.$$

The Frenet-Serret Apparatus

The **Frenet-Serret apparatus** of the unit speed curve $\alpha(s)$ is

$$\{\kappa(s), \tau(s), \mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}.$$

If $\beta(t)$ is a regular curve we may reparametrize with $t = t(s)$ or $s = s(t)$. Let $\alpha(s) = \beta(t(s))$ be a unit-speed reparametrization of β

and let $\{\kappa(s), \tau(s), \mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ be the Frenet-Serret apparatus of α. Then the Frenet-Serret apparatus of β is

$$\{\kappa(s(t)), \tau(s(t)), \mathbf{T}(s(t)), \mathbf{N}(s(t)), \mathbf{B}(s(t))\}.$$

The set $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ is sometimes called a **moving frame** or **moving trihedron**.

2.4 The Frenet-Serret Theorem

The Frenet-Serret apparatus gives us a great deal of information. This section gives us some tools to understand that information.

We begin with a lemma from linear algebra; if $E = \{\mathbf{e}_i\}$ is a collection of n orthonormal vectors in an n-dimensional inner product space V, then E is a basis for V. Additionally, for any $\mathbf{v} \in V$, we must have $\mathbf{v} = \sum \langle \mathbf{e}_i, \mathbf{v} \rangle \mathbf{e}_i$. This tells us exactly *how* to determine a linear combination to construct any vector in a vector space using only basis vectors.

Frenet-Serret Theorem

The **Frenet-Serret theorem** states that if $\alpha(s)$ is a unit speed curve with $\kappa(s) \neq 0$ and Frenet-Serret appartus $\{\kappa(s), \tau(s), \mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ then:

(a) $\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$

(b) $\mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s)$

(c) $\mathbf{B}'(s) = -\tau(s)\mathbf{N}(s)$

for each s. It is often easier to remember the following:

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}.$$

If $\alpha(s)$ is a unit speed curve with nonzero curvature, then if any of the following are true, then all are true:

(a) α is a plane curve;

(b) **B** is a constant vector;

(c) $\tau(s) = 0$ for all s.

The **osculating plane** to a unit speed curve at the point $\alpha(s)$ is the plane through $\alpha(s)$ perpendicular to **B** (and hence spanned by **T** and **N**). The **normal plane** is the plane through $\alpha(s)$ perpendicular to **T**. The **rectifying plane** is the plane through $\alpha(s)$ perpendicular to **N**. Many of our observations of space curves deal with their projections onto the above three planes.

Helix

A common curve we will use is the **helix**, a regular curve such that for some fixed unit vector **u**, $\langle \mathbf{T}, \mathbf{u} \rangle$ is constant. **u** is the **axis** of the helix. As a corollary, a unit speed curve α with $\kappa \neq 0$ is a helix if and only if there exists a c such that $\tau = c\kappa$.

We conclude this section by listing several useful corollaries:

1. $\alpha(s)$ is a straight line if and only if there is a point $\mathbf{x}_0 \in \mathbb{R}^3$ such that every tangent line through α goes through $\mathbf{x}_0$;

2. Let $\alpha(s)$ be a unit speed curve with $\kappa \neq 0$. Then $\alpha(s)$ lies in a plane if and only if all osculating planes are parallel;

3. Let $\alpha(s)$ be a unit speed curve whose image lies on a sphere with radius r and center m. Then $\kappa \neq 0$. If $\tau \neq 0$ then

$$\alpha - m = -\rho\mathbf{N} - \rho'\sigma\mathbf{B}$$

   where $\rho = 1/\kappa$ and $\sigma = 1/\tau$. Hence $r^2 = \rho^2 + (\rho'\sigma)^2$. ρ is the **radius of curvature** and σ is the **radius of torsion**.

2.5 Picard's Existence Theorem and The Fundamental Theorem of Curves

Think back to middle school geometry – we began by describing a triangle by the lengths of all three of its sides and all three of its angles. We quickly learened an abbreviated method – we could determine all the properties of a triangle just by knowing the lengths of two sides and the angle between them. Similarly, we can entirely describe the geometry of a curve using only its torsion and curvature.

Picard's Existence Theorem

Suppose that the $\mathbb{R}^n$-valued function $\mathbf{A}(\mathbf{x}, t)$ is defined and continuous in the closed region $|\mathbf{x} - \mathbf{c}| \leq K$, $|t - a| \leq T$, and satisfies a Lipschitz condition there. Let $M = \sup |\mathbf{A}(\mathbf{x}, t)|$ over this region. Then the differential equation

$$\frac{d\alpha}{dt} = \mathbf{A}(\mathbf{x}, t)$$

has a unique solution on $|t - a| \leq \min(T, K/M)$, where $\alpha(a) = \mathbf{c}$.

The Fundamental Theorem of Curves

Any curve with $\kappa > 0$ is completely determined, up to position, by its curvature and torsion. More precisely, let (a, b) be an interval about 0, $\overline{\kappa}(s) > 0$ a C^1 function on (a, b), $\overline{\tau}(s)$ a continuous function on (a, b), $\mathbf{x}_0$ a fixed point of $\mathbb{R}^3$, and $\{\mathbf{D}, \mathbf{E}, \mathbf{F}\}$ a right-handed orthonormal basis for $\mathbb{R}^3$. Then there is a unique C^3 regular curve $\alpha : (a, b) \to \mathbb{R}^3$ such that:

(a) the parameter is arc length from $\alpha(0)$;

(b) $\alpha(0) = \mathbf{x}_0$, $\mathbf{T}(0) = \mathbf{D}$, $\mathbf{N}(0) = \mathbf{E}$, and $\mathbf{B}(0) = \mathbf{F}$;

(c) $\kappa(s) = \overline{\kappa}(s)$ and $\tau(s) = \overline{\tau}(s)$.

Unlike our 8th grade geometry triangle, it's often pretty difficult to find the initial curve α. In the case of a helix, i.e. $\kappa > 0$ and $\tau = c\kappa$, we can (almost) find the equation of the curve $\alpha(s)$ characterized by κ and τ. The curve is:

$$\alpha(s) = \frac{1}{\omega}\left(\int_0^s \sin \omega t(\sigma), -\int_0^s \cos \omega t(\sigma), cs\right) + \alpha(0),$$

Where $\omega = \sqrt{1 + c^2}$, $t(\sigma) = \int_0^\sigma \kappa(s) ds$.

2.6 Non-unit Speed Curves

In practice, it may not be possible to parametrize by arc length; here we talk about how to compute the Frenet-Serret apparatus for non-unit speed curves. Let $\beta(t)$ be regular, and let $s(t)$ be arc length. Then $\beta(t) = \alpha(s(t))$, where $\alpha(s)$ is $\beta(t)$ reparametrized by arc length. We write $\dot{\beta} = d\beta/dt$, $\ddot{\beta} = d^2\beta/dt^2$, and $\dddot{\beta} = d^3\beta/dt^3$.

The Frenet-Serret apparatus for this curve is then given by the following set of equations:

(a) $\mathbf{T} = \dot{\beta}/|\dot{\beta}|$;

(b) $\mathbf{B} = \dot{\beta} \times \ddot{\beta}/|\dot{\beta} \times \ddot{\beta}|$;

(c) $\mathbf{N} = \mathbf{B} \times \mathbf{T}$;

(d) $\kappa = |\dot{\beta} \times \ddot{\beta}|/|\dot{\beta}|^3$; and

(e) $\tau = \langle \dot{\beta} \times \ddot{\beta}, \dddot{\beta} \rangle / |\dot{\beta} \times \ddot{\beta}|^2$.

The modified Frenet-Serret equations are then:

(a) $\dot{\mathbf{T}} = \kappa|\dot{\beta}|\mathbf{N}$

(b) $\dot{\mathbf{N}} = -\kappa|\dot{\beta}|\mathbf{T} + \tau|\dot{\beta}|\mathbf{B}$

(c) $\dot{\mathbf{B}} = -\tau|\dot{\beta}|\mathbf{N}$.

3 The Global Theory of Plane Curves

The last few sections dealt with the behavior of a curve within a small neighborhood of a point, looking at values such as curvature and torsion locally. In this chapter, we take a more macroscopic/global approach and examine the properties of the entire curve. Much of this will be review from Calculus III.

3.1 Line Integrals and Green's Theorem

Line Integral

Suppose $\alpha(t) = (x(t), y(t))$ is a C^1 parametrization of the geometric curve $\mathscr{C}$ in $\mathbb{R}^2$ over the interval $a \leq t \leq b$. The **line integral** for real-valued functions f, g is then

$$\int_{\mathscr{C}} f dx + g dy = \int_a^b \left[f(x(t), y(t)) \frac{dx}{dt} + g(x(t), y(t)) \frac{dy}{dt} \right] dt.$$

The line integral over a closed loop is denoted

$$\oint_{\mathscr{C}} f dx + g dy.$$

As an example, let $\mathscr{C}$ be the unit circle parametrized by $\alpha(t) = (\cos(t), \sin(t))$ for $0 \leq t \leq 2\pi$. Then

$$\oint_{\mathscr{C}} xdy - ydx = \int_0^{2\pi} \cos^2\theta + \sin^2\theta d\theta = \int_0^{2\pi} d\theta = 2\pi.$$

Line integration is assumed to be with respect to arc length; if the function is not parametrized by arc length, it must be reparametrized. Suppose we cannot reparametrize – for example, say we want $\int_{\mathscr{C}} \kappa(s)dt$, where $s(t)$ is not computable. Then we we can instead compute $\int_{\mathscr{C}} \kappa(s)(ds/dt)dt$, where $\mathscr{C}$ is parametrized by t.

Green's Theorem

If $\mathscr{C}$ is a closed plane curve made up of C^2 curve segments, which bounds a region $\mathscr{R}$, traversed counterclockwise, then

$$\oint_{\mathscr{C}} fdx + gdy = \iint_{\mathscr{R}} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy,$$

for all differentiable f, g defined on $\mathscr{R}$.

3.2 The Rotation Index of Plane Curves

Suppose $\alpha : (a, b) \to \mathbb{R}^3$ is a plane curve (so that we may equivalently state that $\alpha : (a, b) \to \mathbb{R}^2$). This allows us to define a globally consistent normal vector field, as opposed to our 3-D version which requires curvature to be nonzero.

Let α be a unit speed curve in $\mathbb{C}^2$. The **tangent vector field** is $\mathbf{t}(s) = \alpha'(s)$. The **normal vector field** is the unique unit vector field $\mathbf{n}(s)$ such that $\{\mathbf{t}(s), \mathbf{n}(s)\}$ gives a right-handed orthonormal basis of $\mathbb{R}^2$. The **plane curvature** $k(s)$ of α is given by $k(s) = \langle \mathbf{t}'(s), \mathbf{n}(s)\rangle$. Furthermore:

(a) $\mathbf{t}'(s) = k(s)\mathbf{n}(s)$;

(b) If $\alpha(s) = (x(s), y(s))$ then

$$\mathbf{t}(s) = (x'(s), y'(s)), \quad \mathbf{n}(s) = (-y'(s), x'(s)).$$

We can additionally relate these definitions to our 3-D Frenet-Serret apparatus; $\mathbf{t}(s) = \mathbf{T}(s)$ and $\mathbf{n}(s) = \pm\mathbf{N}(s)$ (wherever $\mathbf{N}(s)$ is defined), $\kappa(s) =$

$|k(s)|$, $\mathbf{n}(s)$ is differentiable with $\mathbf{n}(s) = -k(s)\mathbf{t}(s)$. The sign of k indicates whether the curve is curving away from or towards $\mathbf{n}$.

A regular curve $\beta(t)$ is **closed** if β is **periodic**, meaning that for some fixed constant $a > 0$, $\beta(t) = \beta(t+a)$ for all t. The smallest such a is the **period** of β. If $\alpha(s)$ is an arc-length parametrization of β, it is also closed, and its period is

$$L = \int_0^a \left| \frac{d\beta}{dt} \right| dt.$$

$\beta(t)$ is **simple** if β is not self-intersecting; more formally, if either β is injective or if $\beta(t_1) = \beta(t_2)$ if and only if $t_2 - t_1 = na$ for $n \in \mathbb{Z}$.

Rotation Index Theorem

An important property that we can now examine is the **rotation index** or **winding number** of a curve. This value represents how many full counterclockwise rotations a particle would make in one period with respect to a point, and is a fundamental object of study in algebraic topology, complex analysis, geometric topology, and string theory.

The **rotation index** of a closed unit speed plane curve is the integer

$$i_\alpha = \frac{\theta(L)}{2\pi},$$

where $\theta(s)$ is a continuous function describing the angle of the tangent vector. $\alpha(s)$ is assumed to be oriented in the top half of the plane such that $\theta(0) = 0$. The rotation index of a simple closed plane curve is ± 1.

If $\alpha(s)$ is a simple closed regular plane curve, the **tangent circular image** $\mathbf{t} : [0, L] \to \mathbb{S}^1$ (where $\mathbb{S}^1$ is the unit circle in the plane) is surjective.

3.3 Convex Curves

A line l divides the 2-D plane $\mathbb{R}^2$ into two **half-planes**, denoted H_1 and H_2, such that $H_1 \cup H_2 = \mathbb{R}^2$ and $H_1 \cap H_2 = l$. A curve **lies on one side of** a line l if every point in its image is contained entirely within one half plane.

Convexity

A regular curve α is **convex** if it lies on one side of every tangent line to α.

A function f is **monotonically increasing** if $s \leq t$ implies $f(s) \leq f(t)$; a function is **monotonically decreasing** if $s \leq t$ implies $f(s) \geq f(t)$. This in turn implies that a simple closed regular plane curve $\alpha(s)$ is convex if and only if $k(s)$ has constant sign (this would imply that the normal vector is pointing away from the "interior" of the curve, or that the angle $\theta(s)$ is monotonic).

The natural implication of this is if we have a convex curve with the property that for $s_1 \neq s_2$ we have $\theta(s_1) = \theta(s_2)$, the curve must be a straight line segment between s_1 and s_2. In fact, if a line l passes through 3 points of a convex curve, then the entire line segment connecting those 3 points must be in the image of α.

3.4 The Isoperimetric Inequality

From elementary geometry we know that, for a fixed perimeter, the circle is the shape with the greatest enclosed area. Here we establish that this is true for *all* simple closed plane curves.

The Isoperimetric Inequality

If α is a simple closed plane curve enclosing a region $\mathscr{R}$, the area of $\mathscr{R}$ is

$$\oint_\alpha x dy = -\oint_\alpha y dx$$

where x and y are the coordinates of the plane. Let the length of α be L. Let the area bounded by α be A. Then

$$L^2 \geq 4\pi A$$

with equality only if α is a circle. Thus, of all curves with length L, the circle bounds the greatest area.

3.5 The Four-Vertex Theorem

This section deals with the special class of convex curves with no straight segments or isolated points where $k = 0$. An **oval** is a regular simple convex closed plane curve with $k > 0$. A **vertex** of a regular plane curve is a point where k has a relative maximum or minimum.

The Four-Vertex Theorem

An oval $\alpha(s)$ has at least four vertices. In fact, this is true for any simple closed plane curve.

Let P be a point on the curve α, where α is an oval. Since the tangent circular image is surjective, there must be a point $\bar{P}$ where $\mathbf{t}(P) = -\mathbf{t}(\bar{P})$. Therefore the tangent lines at P and $\bar{P}$ must be parallel. The **width** is the perpendicular distance between those two tangent lines. An oval has **constant width** if the width is independent of the choice of P. A circle is an example of an oval of constant width.

Barbier's Theorem (1860)

If α is an oval of constant width w, its length is πw. Additionally, the straight line joining P and $\bar{P}$ must be orthogonal to the tangents at P and $\bar{P}$.

4 Local Surface Theory

The previous section deals with the geometry of curves, which is fairly intuitive; it's easy to picture what the "geometry" of a curve means, and the topic lends itself nicely to our discussion of torsion and curvature. Here we turn our attention to the study of **surfaces**, a much deeper and complex subject.

4.1 Basic Definitions and Examples

In 2-D geometry, we have a very natural way to "regularize" our curves, since every curve has an arc-length parametrization. Surfaces cannot be regularized in this way. For example, consider the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. No matter what parametrization we choose, there will be (at least) one point which is ambiguous; the poles of a globe cannot be defined by a single latitude-longitude pair.

A subset $\mathcal{U} \subset \mathbb{R}^2$ is **open** if, for every $(a, b) \in \mathcal{U}$, there is a $\varepsilon > 0$ such that $(x, y) \in \mathcal{U}$ whenever

$$(x-a)^2 + (y-b)^2 < \varepsilon^2.$$

Essentially, $\mathcal{U}$ is open if there is an adequately small disc about each point in $\mathcal{U}$ which is also contained entirely in $\mathcal{U}$.

A C^k **coordinate patch** (a **simple surface**) is an injective C^k function $\mathbf{x} : \mathcal{U} \to \mathbb{R}^3$ for some $k \geq 1$ where $\mathcal{U}$ is an open subset of $\mathbb{R}^2$ with coordinates u^1, u^2 and $(\partial \mathbf{x}/\partial u^1) \times (\partial \mathbf{x}/\partial u^2) \neq 0$ on $\mathcal{U}$. A simple surface is a 3-D function which is injective with respect to its 2-D "shadow". Our condition, that the cross product of partial derivatives with respect to the coordinates of the "shadow," is equivalent to our condition earlier of regularity; in fact, we call this the **regularity condition for surfaces.** This condition, that $(\partial \mathbf{x}/\partial u^1) \times (\partial \mathbf{x}/\partial u^2) \neq 0$, is equivalent to requiring that $\{\partial \mathbf{x}/\partial u^1, \partial \mathbf{x}/\partial u^2\}$ is a linearly independent set.

Take, as an example, $f(u^1, u^2)$, a C^k differentiable function in an open set $\mathcal{U}$. Let $\mathbf{x} = (u^1, u^2, f(u^1, u^2))$. $\mathbf{x}$ is a C^k simple surface which is the graph of a function, and is called a **Monge patch**.

A C^k **coordinate transform** (also called a **diffeomorphism**) is a C^k injective function $f : \mathcal{V} \to \mathcal{U}$ of open sets in $\mathbb{R}^2$ whose inverse $g : \mathcal{U} \to \mathcal{V}$ is also of class C^k. If $\mathbf{x} : \mathcal{U} \to \mathbb{R}^3$ is a simple surface and $f : \mathcal{V} \to \mathcal{U}$ is a coordinate transform, then $\mathbf{y} = \mathbf{x} \circ f : \mathcal{V} \to \mathbb{R}^3$ is a simple surface with the same image as $\mathbf{x}$. For brevity of notation, we use the following shorthand for $\mathbf{x} : \mathcal{U} \to \mathbb{R}^3$ a simple surface:

$$\mathbf{x}_1 = \frac{\partial \mathbf{x}}{\partial u^1}, \quad \mathbf{x}_2 = \frac{\partial \mathbf{x}}{\partial u^2}$$

The **tangent plane** to a simple surface at the point $P = \mathbf{x}(a, b)$ is the plane through P perpendicular to $\mathbf{x}_1(a, b) \times \mathbf{x}_2(a, b)$. The **unit normal** to the surface at P is $\mathbf{n}(a, b) = \mathbf{x}_1 \times \mathbf{x}_2 / |\mathbf{x}_1 \times \mathbf{x}_2|$, where the RHS is evaluated at (a, b). The normal plane always exists due to our regularity condition, and is perpendicular to the tangent plane. The tangent and normal planes are preserved through coordinate transform, except the normal plane may have the opposite sign. The coordinate transform in this way is analogous to the concept of reparametrization. A **tangent vector** to a surface $\mathbf{x}$ at

P is the velocity vector of some curve in $\mathbf{x}$ passing through P; the set of all such velocity vectors forms a vector space.

4.2 Surfaces

Our discussion so far is unable to characterize the entire sphere $\mathbb{S}^2$. To do this we introduce a more general notion of a **surface** – a collection of overlapping simple surfaces.

Let M be a subset of $\mathbb{R}^3$ and let $\varepsilon > 0$. The ε-neighborhood of $P \in M$ is the set of all points $Q \in M$ such that $d(P, Q) < \varepsilon$ where d is standard Euclidean distance. If $M \subset \mathbb{R}^3$, the function $g : M \to \mathbb{R}^2$ is **continuous** at $P \in M$ if for every open set $\mathcal{U}$ in $\mathbb{R}^2$ with $g(P) \in \mathcal{U}$ there is an $\varepsilon-$neighborhood $\mathcal{N}$ of P with $g(\mathcal{N}) \subset \mathcal{U}$. A coordinate patch is **proper** if the inverse function $\mathbf{x}^{-1} : \mathbf{x}(\mathcal{U}) \to \mathcal{U}$ is continuous at each point of $\mathbf{x}(\mathcal{U})$.

Surface

A C^k **surface** in $\mathbb{R}^3$ is a subset $M \subset \mathbb{R}^3$ such that for every point $P \in M$ there is a proper C^k coordinate patch whose image is in M and which contains an $\varepsilon-$neighborhood of P for some $\varepsilon > 0$. Furthermore, if both $\mathbf{x} : \mathcal{U} \to \mathbb{R}^3$ and $\mathbf{y} : \mathcal{V} \to \mathbb{R}^3$ are such coordinate patches with $\mathcal{U}' = \mathbf{x}(\mathcal{U})$, $\mathcal{V}' = \mathbf{x}(\mathcal{V})$, then $y^{-1} \circ \mathbf{x} : (x^{-1}(\mathcal{U}' \cap \mathcal{V}')) \to (y^{-1}(\mathcal{U}' \cap \mathcal{V}'))$ is a C^k coordinate transformation. A sphere is a surface as it can be completely covered by 6 coordinate patches – the left and right hemispheres, the top and bottom hemispheres, and the front and back hemispheres.

The **implicit function theorem** states that if $f : \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function such that $(f_x, f_y, f_z) \neq 0$ at all points $M = \{(x, y, z) : f(x, y, z) = 0\}$ then M is a surface, and if $f_z \neq 0$ at $P \in M$ then there is a Monge patch in M that contains P.

4.3 The First Fundamental Form and Arc Length

There is a problem when we examine coordinate patches independently. Any definition in one coordinate patch must be manually checked with its form in another coordinate patch to check if that definition is geometric. Let M be a surface in $\mathbb{R}^3$ and $P \in M$. If $\mathbf{X}$ and $\mathbf{Y}$ are vectors tangent to M at P, we would like to compute the inner product

$$\langle \mathbf{X}, \mathbf{Y} \rangle = \sum X_i Y_j g_{ij},$$

where

$$g_{ij}(u^1, u^2) = \langle \mathbf{x}_i(u^1, u^2), \mathbf{x}_j(u^1, u^2) \rangle.$$

g defines a symmetric matrix, a function defined on $\mathcal{U}$, known as the **metric coefficients**, the **metric tensor coefficients**, or the **Riemannian metric coefficients**.

The **tangent space** of a surface M at $P \in M$ is the set $T_P M$ of all vectors tangent to M at P. This is the same as the tangent plane at P to *any* of the coordinate patches containing P. If we then restrict our inner product $\langle , \rangle$ to this space, (g_{ij}) is a representation of the restricted inner product with respect to the basis $\{\mathbf{x}_1, \mathbf{x}_2\}$, and (g_{ij}) is a nonsingular positive definite matrix. The rule assigning an inner product to tangent vectors $\mathbf{X}, \mathbf{Y} \in T_P M$ is known as the **first fundamental form**. denoted

$$\mathbf{I}(\mathbf{X}, \mathbf{Y}).$$

We use the "upper index" notation g^{kl} to represent the k, l entry of the inverse matrix of (g_{ij}). For a coordinate patch $\mathbf{x} : \mathcal{U} \to \mathbb{R}^3$:

(a) $\det(g) = |\mathbf{x}_1 \times \mathbf{x}_2|^2$;

(b) $g^{1,1} = g_{2,2}/\det(g)$, $g^{1,2} = g^{2,1} = -g_{1,2}/\det(g)$, $g^{2,2} = g_{1,1}/\det(g)$;

(c) for all i, j: $\sum_{k=1}^{2} g_{i,k} g^{k,j} = \delta_{i,j}$.

Duality

A **linear functional** on a real vector space V is a linear function

$$\rho : V \to \mathbb{R}.$$

The set of all linear functionals defined on V forms a vector space under the usual concepts of addition and scalar multiplication, i.e.

$$(r\varphi + \psi)(\mathbf{v}) = r(\varphi(\mathbf{v})) + \psi(\mathbf{v}).$$

This vector space is the **dual space** of V, denoted V^*, and has the same dimension as V.

4.4 Normal and Geodesic Curvature and Gauss's Formulas

Let $\gamma(s)$ be a unit-speed curve whose images lies on the surface $M \subset \mathbb{R}^3$. γ has Frenet-Serret apparatus $\{\kappa, \tau, \mathbf{T}, \mathbf{N}, \mathbf{B}\}$.

Then if $\mathbf{x} : \mathcal{U} \to \mathbb{R}^3$ is a simple surface and $\gamma(s)$ is a unit-speed curve in the image of $\mathbf{x}$, then

(a) $\mathbf{x}_{i,j} = \frac{\partial^2 \mathbf{x}}{\partial u_j \partial u_i}(a, b)$;

(b) $\mathbf{x}_{i,j} = \frac{\partial^2 \mathbf{x}}{\partial u_j \partial u_i}$;

(c) $\mathbf{S} = \mathbf{n} \times \mathbf{T}$.

$\mathbf{S}$ is the **intrinsic normal** of γ, and is well defined on M up to sign (just like $\mathbf{n}$). Be careful here; $\mathbf{S}$ is normal to the curve itself but is tangent to the surface. If $P \in M$ then the set $N_p M = \{r\mathbf{n} | r \in \mathbb{R}\}$ is the set of all vectors perpendicular to M at P and is called the **normal space** of M at P. $T_P M + N_P M = \mathbb{R}^3$, which means any vector can be decomposed into the sum of a tangent and a normal vector.

Now let $\gamma'' = \mathbf{X}(s) + \mathbf{V}(s)$ where $\mathbf{X}$ is tangent to M and $\mathbf{V}$ is normal to M. Then $\mathbf{T}$ is normal to $\mathbf{V}$ so $\langle \mathbf{T}, \mathbf{V} \rangle = 0$; since $\langle \gamma'', \mathbf{T} \rangle = 0$ as well, $\langle \mathbf{X}, \mathbf{T} \rangle$ must also be 0. Then $\mathbf{X}$ is a multiple of $\mathbf{S}$.

Normal and Geodesic Curvature

Define two functions:

$$\kappa_n(s) = \langle \gamma''(s), \mathbf{n}(\gamma^1(s), \gamma^2(s)) \rangle$$

$$\kappa_g(s) = \langle \gamma''(s), \mathbf{S}(s) \rangle$$

so that

$$\kappa(s)\mathbf{N}(s) = \mathbf{T}'(s) = \gamma''(s) = \kappa_n(s)\mathbf{n}(s) + \kappa_g(s)\mathbf{S}(s).$$

The normal component of γ'', $\kappa_n(s)$, is the **normal curvature** of the unit speed curve γ. The component in the direction of $\mathbf{S}$, $\kappa_g(s)$, is the **geodesic curvature** of γ.

The coefficients of the **second fundamental form** of a simple surface $\mathbf{x} : \mathcal{U} \to \mathbb{R}^3$ are the functions L_{ij} defined on $\mathcal{U}$ by $L_{ij} = \langle \mathbf{x}_{ij}, \mathbf{n} \rangle$. The

Christoffel symbols are the functions $\Gamma^k_{ij}(1 \leq i, j, k \leq 2)$ defined on $\mathcal{U}$ by

$$\Gamma^k_{ij} = \sum_{l=1}^{2} \langle \mathbf{x}_{ij}, \mathbf{x}_l \rangle g^{lk}.$$

Since (L_{ij}) is a symmetric matrix, (Γ^k_{ij}) is as well. The L_{ij} coefficients measure the normal component of $\mathbf{x}_{ij}$ while Γ^k_{ij} measures the tangential components.

Gauss's Formulas

Let $\mathbf{x} : \mathcal{U} \to \mathbb{R}^3$ be a simple surface. Then the following formulas hold:

(a) $\mathbf{x}_{ij} = L_{ij}\mathbf{n} + \sum_k \Gamma^k_{ij}\mathbf{x}_k$.

(b) For a unit speed curve $\gamma(s) = \mathbf{x}(\gamma^1(s), \gamma^2(s))$,

$$\kappa_n = \sum_{i,j} L_{ij}(\gamma^1)'(\gamma^2)'$$

(c) $\kappa_g \mathbf{S} = \sum_k [(\gamma^k)'' + \sum_{ij} \Gamma^k_{ij}(\gamma^i)'(\gamma^j)']\mathbf{x}_k$.

The Christoffel coefficients are **intrinsic**, meaning they can be determined by measurements within the surface (meaning they only depend on (g_{ij})), as is the geodesic curvature. A two-dimensional being would only have concepts of lengths and angles and the derived metric coefficients, and those would be the only geometric concepts they know. Something like the normal vector to the surface would be incomprehensible.

The Levi-Civita Connection

From Gauss's formulas we obtain that $\langle \mathbf{x}_{ij}, \mathbf{x}l \rangle = \sum_k \Gamma^k_{ij} g_{kl}$. We write this quantity as $\Gamma_{ij|l}$ and is called a **Christoffel symbol of the first kind**. The traditional Γ^k_{ij} is a **Christoffel symbol of the second kind**. For a coordinate patch $\mathbf{x} : \mathcal{U} \to \mathbb{R}^3$ with metric coefficients g_{ij}

$$\Gamma^l_{ij} = \frac{1}{2} \sum_{k=1}^{2} g^{kl} \left(\frac{\partial g_{ik}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k} + \frac{\partial g_{kj}}{\partial u^i} \right).$$

This relationship outlines a **connection** – an affine transformation between tangent spaces. This particular connection which is torsion-free and preserves the metric tensor is known as the **Levi-Civita connection** and is closely related to the **Levi-Civita parallel transport** discussed in section 4.6.

4.5 Geodesics

In our normal multivariate plane geometry, we rely heavily on the **straight line**. Now that we're looking at arbitrary surfaces, we want to define curves that serve similar roles. Straight lines have several important properties, and we need to choose how to represent those properties on an arbitrary surface.

1. Straight lines have zero plane curvature;
2. Straight lines give the shortest path between two points;
3. Any two points have a unique straight line joining them;
4. All tangent vectors to a straight line are parallel.

We can find a general solution that will have properties (1) and (2), but not (3); (3) is not true in general, as it makes assumptions about the topological properties of the surface, such as simple connectivity. The next section will focus on property (4).

Geodesic

A **geodesic** on a surface M is a unit-speed curve on M with geodesic curvature equal to zero everywhere; formally, $\gamma(s) \in M$ is a geodesic if and only if $\langle \mathbf{n} \times \mathbf{T}, \mathbf{T}' \rangle = 0$. If γ is the unit speed curve $\gamma(s) = \mathbf{x}(\gamma^1(s), \gamma^2(s))$ on a coordinate patch $\mathbf{x}$, then γ is a geodesic if and only if

$$\gamma^{k\prime\prime} + \sum \Gamma^k_{ij} \gamma^{i\prime} \gamma^{j\prime} = 0.$$

More generally, a unit speed curve γ on a surface M is a geodesic if and only if γ'' is everywhere normal to the surface.

This immediately satisfies property (1).

We can define the geodesics for a surface of revolution. Let M be the surface of revolution generated by the unit speed curve $(r(t), z(t))$. Then every meridian is a geodesic, and a circle of latitude is a geodesic if and only if the tangent $\mathbf{x}_1$ to the meridians is parallel to the axis of revolution at all points on the circle of latitude.

We can determine a straight line given a point and a direction at that point. The same is generally true for geodesics. Let P be a point on a surface M and let $\mathbf{X}$ be a unit tangent vector at P. Then if $s_0 \in \mathbb{R}$ is given, there exists a unique geodesic γ with $\gamma(s_0) = P$, $\gamma'(s_0) = \mathbf{X}$. A surface is **complete** if every geodesic extends indefinitely; M is complete if and only if it is complete as a metric space.

Property (2) is also satisfied; the proof is involved and is a cornerstone of the calculus of variations, but is omitted here for brevity. Let γ be a unit speed curve between points $P = \gamma(a)$ and $Q = \gamma(b)$. If γ is the shortest curve between P and Q, then it is a geodesic.

4.6 Parallelism

Here we talk about Property (4), attempting to generalize the notion of "parallel." More specifically, we want to generalize the notion of parallel *vectors* as opposed to parallel lines; in general, the notion of parallel lines (as can be seen on a sphere) do not exist.

A **vector field along a curve** $\gamma : [a, b] \to M$ is a function $\mathbf{X}$ which assigns to each $t \in [a, b]$ a tangent vector $\mathbf{X}(t)$ to M at $\gamma(t)$. A differentiable vector field $\mathbf{X}(t)$ along γ is **parallel** along γ if $d\mathbf{X}/dt$ is perpendicular to M. Equivalently:

$$0 = \frac{dX^k}{dt} + \sum \Gamma^k_{ij} X^i \frac{d\gamma^j}{dt}.$$

This characterization is intrinsic (independent of parametrization).

Now let $\tilde{\mathbf{X}}$ be a vector tangent to M at $\gamma(t_0)$. Then there exists a *unique* vector field $\mathbf{X}(t)$ that is parallel along $\gamma(t)$ with $\mathbf{X}(t_0) = \tilde{\mathbf{X}}$. This conclusion follows from turning the previous equation into an initial-value system of differential equations:

$$\begin{aligned} \frac{dX^k}{dt} &= -\sum \Gamma^k_{ij}(\gamma^1(t), \gamma^2(t)) X^i(t) \frac{d\gamma^j}{dt}, \\ X^k(t_0) &= \tilde{X}^k. \end{aligned}$$

The vector field $\mathbf{X}$ in the previous theorem is called the **parallel translate**, **parallel transport**, or **Levi-Cevita connection** (for general manifolds) of $\tilde{\mathbf{X}}$ along γ. Just as in plane geometry, the parallel translate preserves the angle and length of vectors; however, if two *different* curves connect P and Q, the vector field at Q depends on which curve is used.

Given this notion of "parallel," we say that a regular curve γ is **maximally straight** if $d\gamma/dt$ is parallel along γ. This happens only when dt/ds is constant and $\gamma(t(s))$ is a geodesic. This completes our generalization of Property (4).

4.7 The Second Fundamental Form and the Weingarten Map

The previous sections revolved around the geodesic curvature; here we consider the normal curvature, a measure of how M is curving in the direction of $\mathbf{T}$; if we just want to measure how M is curving without direction, we introduce the Weingarten map L.

The **second fundamental form II** on M is the bilinear form on $T_P M$ for each point $P \in M$ given by

$$\mathbf{II}(\mathbf{X}, \mathbf{Y}) = \sum_{ij} L_{ij} X^i Y^j$$

where

$$\mathbf{X} = \sum X^i \mathbf{x}_i, \quad \mathbf{Y} = \sum Y^i \mathbf{x}_j$$

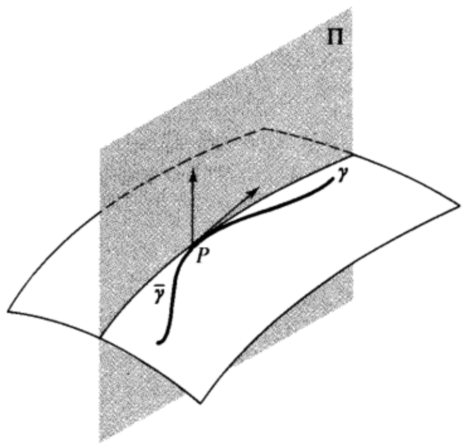
and L_{ij} are the coefficients $\langle \mathbf{x}_{ij}, \mathbf{n} \rangle$.

(a) $\mathbf{II}$ is a symmetric bilinear form on $T_P M$;

(b) If γ is a unit speed curve with tangent $\mathbf{T}$, then $\kappa_n = \mathbf{II}(\mathbf{T}, \mathbf{T})$;

(c) If α and β are regular curves with $\alpha(0) = \beta(0)$ and whose velocity vectors are dependent at $t = 0$, then α and β have the same normal curvature at $t = 0$.

Let $\gamma(s)$ be a unit speed curve in a surface M with normal curvature κ_n at P. Let $\bar{\gamma}$ be the curved formed by the intersection of M with the plane Π through P spanned by $\mathbf{n}$ and γ'. Then $|\kappa_n|$ is the curvature $\bar{\kappa}$ of the plane curve $\bar{\gamma}$.



A subset $\mathscr{R}$ of M is **open** if for each $P \in \mathscr{R}$ there is an ε-neighborhood of P in M contained in $\mathscr{R}$. A function $f : \mathscr{R} \to \mathbb{R}$ is **differentiable** if for every C^1 curve $\alpha(t)$ with $\alpha(0) \in \mathscr{R}$ the derivative $(d(f \circ \alpha)/dt)(0)$ exists. Let $\alpha(0) = P$ and let $\mathbf{X} \in T_P M$ be $\mathbf{X} = (d\alpha/dt)(0)$. The **directional derivative** of f in the direction of $\mathbf{X}$ is $\mathbf{X}f = (d(f \circ \alpha)/dt)(0)$. $\mathbf{X}f$ is well-defined and is independent of our choice of α.

Let $\mathbf{x} : \mathcal{U} \to M$ be a coordinate patch for M about $P = \mathbf{x}(0,0)$. If $\mathbf{X} = \sum X_i \mathbf{x}_i$ then $\mathbf{X}f = \sum_{i=1}^{2} X_i(\partial(f \circ \mathbf{x})/\partial u_i)(0,0)$. The directional derivative is linear, meaning that for $\mathbf{X}, \mathbf{Y} \in T_P M$,

$$(\alpha\mathbf{X} + \beta\mathbf{Y})f = \alpha(\mathbf{X}f) + \beta(\mathbf{Y}f).$$

The Weingarten Map

The **Weingarten map** L is, for each $P \in M$, the function $\mathsf{L} : T_P M \to \mathbb{R}^3$ given by

$$\mathsf{L}(\mathbf{X}) = -\mathbf{X}\mathbf{n}.$$

$\mathbf{n}$ is only determined up to sign, so so is L. L is a linear transformation from $T_P M$ to $T_P M$. L is a **self-adjoint** (symmetric) linear transformation, i.e.

$$\mathbf{II}(\mathbf{X}, \mathbf{Y}) = \langle \mathsf{L}(\mathbf{X}), \mathbf{Y} \rangle = \langle \mathbf{X}, \mathsf{L}(\mathbf{Y}) \rangle.$$

Weingarten Equations

L_k^l is the representation of L with respect to the basis $\{\mathbf{x}_1, \mathbf{x}_2\}$. If $\mathsf{L}(\mathbf{x}_k) = \sum L_k^l \mathbf{x}_l$, then $L_k^l = \sum L_{ik} g^{il}$. We say that L_k^l is obtained from L_{ik} by **raising an index** to a tensor of type $(1,1)$ because it has one upper and one lower index. We then get **Weingarten's equations** for a surface M, that

$$\mathbf{n}_j = -\sum L_j^k \mathbf{x}_k.$$

4.8 Principal, Gaussian, Mean, and Normal Curvature

At this point we have several methods to determine how a surface curves. We can use the normal curvature of curves, but this is unappealing as it involves splitting a surface into infinitely many curves. We also have the Weingarten map L, which tells us how the normal vector changes along the surface. This section discusses two more types of curvature that arise naturally from the linearity of L.

If we knew all possible values of κ_n at P, we would now how M curves. The first step is to find the minimum and maximum values of κ_n, which we can do by finding the max and min of $\mathbf{II}(\mathbf{X}, \mathbf{X})$ as $\mathbf{X}$ runs over $T_P M$. We can use the Lagrange multiplier and find the critical values of

$$f(\mathbf{X}, \lambda) = \mathbf{II}(\mathbf{X}, \mathbf{X}) - \lambda(\langle \mathbf{X}, \mathbf{X} \rangle - 1).$$

Evaluation yields the relation that

$$\mathsf{L}(\mathbf{X}) = \lambda \mathbf{X},$$

implying that λ is an eigenvalue of L and $\mathbf{X}$ is the corresponding eigenvector. The self-adjointness of L means that real values of λ exist; the max and min must exist since the set of unit vectors in $T_P M$ is compact.

Principal Curvature

The eigenvalues of L are the roots of the characteristic equation

$$0 = \lambda^2 - (\mathbf{tr}(\mathsf{L}))\lambda + \det \mathsf{L}.$$

If $\lambda, \mathbf{X}$ is an eigenvalue-eigenvector pair, and $\mathbf{Y} \in T_P M$ is unit-length such that $\langle \mathbf{X}, \mathbf{Y} \rangle = 0$, then $\mathbf{Y}$ is also an eigenvector. In fact, there exist orthogonal directions such that the normal curvature κ_n attains its maximum in one direction and minimum in the other.

The **principal curvatures** of M at a point P are the eigenvalues of L, denoted κ_1 and κ_2 where $\kappa_1 \geq \kappa_2$ by convention. The corresponding eigenvectors are the **principal directions** of M at P. If $\kappa_1 = \kappa_2$, we call the corresponding point P an **umbilical point**; a point at which the surface is locally spherical. A **line of curvature** on a surface M is a curve whose tangent vector at each point is a principal direction at that point.

Euler's Curvature Theorem

Let $\mathbf{Y}$ be a unit vector tangent to M at P. Then

$$\mathbf{II}(\mathbf{Y}, \mathbf{Y}) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta,$$

where θ is the angle between $\mathbf{Y}$ and the principal direction corresponding to κ_1.

Gaussian and Mean Curvature

The **Gaussian curvature** of M at P is $K = \kappa_1\kappa_2 = \det \mathsf{L}$. The **mean curvature** is $H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2}\mathbf{tr}(\mathsf{L})$. H is the average normal curvature; surfaces with $H = 0$ are called **minimal surfaces**.

Gaussian curvature is difficult to understand geometrically. To interpret it, we need to understand the concept of surface integration. If $x : \mathcal{U} \to \mathbb{R}^3$ is a parametrized surface the **area of a subset** $\mathscr{R}$ is

$$A(\mathscr{R}) = \iint_{x^{-1}(\mathscr{R})} (\mathbf{x}_1 \times \mathbf{x}_2) \cdot \mathbf{n} du^1 du^2.$$

The **normal spherical image** or **Gauss map** of a surface is a function $v : M \to \mathbb{S}^2$ which sends each point of M to its normal. The Gaussian curvature K at P is the limit of the ratio $A(v(\mathscr{R}))/A(\mathscr{R})$ as $\mathscr{R}$ shrinks to P, where A is signed area.

Dupin Indicatrix

The **Dupin indicatrix** D of M at $P \in M$ is the subset of T_PM given by

$$D = \{\mathbf{X} \in T_PM | \mathbf{II}(\mathbf{X}, \mathbf{X}) = 1\} \cup \{\mathbf{X} \in T_PM | \mathbf{II}(\mathbf{X}, \mathbf{X}) = -1\}.$$

We typically write this as $D = D^+ \cup D^-$. Locally, D^+ and D^- are (possibly degenerate) conic sections – an ellipse (**synclastic**, $K > 0$), a conjugate pair of hyperbolas (**anticlastic**, $K < 0$), two parallel lines (**monoclastic**, $K = 0$), or empty.

Nonzero vectors $\mathbf{X}$, $\mathbf{Y}$ are **conjugate directions** if $\mathbf{II}(\mathbf{X}, \mathbf{Y}) = 0$. A tangent vector $\mathbf{X}$ at P is **asymptotic** if $\mathbf{II}(\mathbf{X}, \mathbf{X}) = 0$. An asymptotic curve is one whose tangent vector is an asymptotic direction at each point.

4.9 Riemannian Curvature Tensor, Gauss's *Theorema Egregium*, and the Codazzi-Mainardi Equations

Riemannian Curvature Tensor

Let $\mathbf{v}x : \mathcal{U} \to \mathbb{R}^3$ be a coordinate patch on M with Christoffel symbols Γ^k_{ij} and second fundamental form coefficients L_{ij}. The **Riemannian curvature tensor** with indices (i, l, j, k) is

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ik}}{\partial u^j} - \frac{\partial \Gamma^l_{ij}}{\partial u^k} + \sum (\Gamma^p_{ik} \Gamma^l_{pj} - \Gamma^p_{ij} \Gamma^l_{pk})$$

for all $1 \leq i, l, j, k \leq 2$.

Of course, this calculation is cumbersome; we usually use the following equations to simplify our calculations. **Gauss's equations** state that

$$R^l_{ijk} = L_{ik} L^l_j - L_{ij} L^l_k,$$

and the **Codazzi-Mainardi equations** further make elucidate that

$$\frac{\partial L_{ij}}{\partial u^k} - \frac{\partial L_{ik}}{\partial u^j} = \sum (\Gamma^l_{ik} L_{ij} - \Gamma^l_{ij} L_{ik}).$$

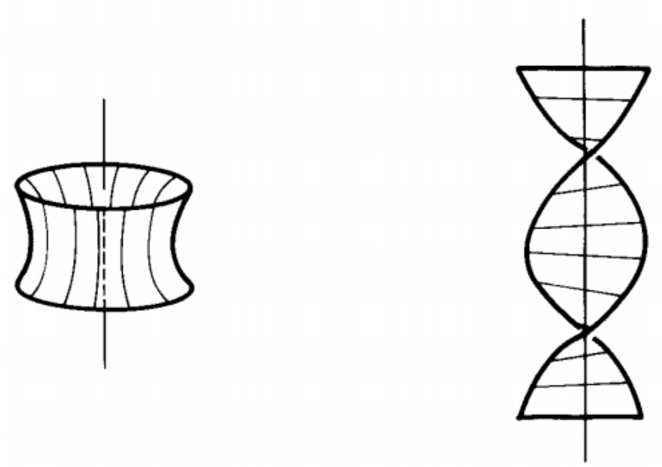
The formal definition provides an intrinsic characterization of the Riemannian curvature tensor; Gauss's equation provides an extrinsic one in terms of the second fundamental form and Weingarten map. From here we get a remarkable result – that even through Gaussian curvature is defined extrinsically (in terms of **n** or L), **the Gaussian curvature K of a surface is an intrinsic property.** This is Gauss's ***Theorema Egrigium***. Perhaps even more astoundingly, this tells us that we can characterize the Gaussian curvature even if we fix all but one of our indices:

$$K = \sum_l R^l_{121} \frac{g_{l2}}{g}$$

The Codazzi-Mainardi equations are important in that they give integrability conditions, as we will see in the next section.

4.10 Isometries and the Fundamental Theorem of Surfaces

Here we examine the question of when two surfaces are geometricallly the same; **isometry** (the same intrinsic geometry) and **rigidity** (intrinsic and extrinsic geometry). A function $f : M \to N$ between surfaces is **differentiable** if for each $P \in M$ there are coordinate patches $\mathbf{x}$ and $\mathbf{y}$ about P and $f(P)$ respectively such that $y^{-1} \circ f \circ \mathbf{x}$ is differentiable as a function of two variables. An **isometry** from M to N is a bijective differentiable function $f : M \to N$ such that for any curve $\gamma : [c, d] \to M$, the length of γ equals the length of $f \circ \gamma$. M and N are **isometric** if such an isometry exists. As an example, the helicoid and catenoid are isometric:



The meridians of the catenoid become the straight lines of a helicoid, and the circles of latitude become circular helices in in the latter.
This particular example might seem similar to Gauss's *Theorema Egrigium*,

and for good reason; **locally isometric surfaces have the same Gaussian curvature at corresponding points**.

We now turn to the stronger concept of rigidity. A **special orthogonal square matrix A** is one with a determinant of 1 with $\mathbf{A}^\top = \mathbf{A}^{-1}$. **A** represents a rotation as a linear transform. $f : \mathbb{R}^n \to \mathbb{R}^n$ is a **rigid motion** if there is a rotation **A** and a vector **b** such that $f(\mathbf{v}) = (\mathbf{A}\mathbf{v}) + \mathbf{b}$. Two surfaces are **rigidly equivalent** if there is a rigid motion f such that $f(M) = N$. In general, surfaces that are isometric need not be rigidly equivalent. Recall that from the Fundamental Theorem of Curves, curvature and torsion determine a space curve up to position – we now say that the space curve is determined up to a rigid motion.

Fundamental Theorem of Surfaces

Let $\mathcal{U}$ be an open set in $\mathbb{R}^2$ such that any two points of $\mathcal{U}$ may be joined by a curve in $\mathcal{U}$ and let $L_{ij} : \mathcal{U} \to \mathbb{R}$ and $g_{ij} : \mathcal{U} \to \mathbb{R}$ be differentiable functions for $i = 1, 2$ and $j = 1, 2$ such that

(a) L and g are symmetric, $g_{11} > 0$, $g_{22} > 0$, and $\det g > 0$, and

(b) L and g satisfy Gauss's equations and the Codazzi Mainardi equations.

Then if $P \in \mathcal{U}$, there is an open set $\mathcal{V} \subset \mathcal{U}$ containing P and a simple surface $\mathbf{v}x$ on $\mathcal{V}$ such that g and L are the matrices of of **I** and **II**, the first and second fundamental forms. Furthermore, if $\mathbf{y} : \mathcal{V} \to \mathbb{R}$ is another simple surface with $\mathbf{I} = g$ and $\mathbf{II} = L$, then $\mathbf{y}(\mathcal{V})$ is rigidly equivalent to $\mathbf{x}(\mathcal{V})$.

4.11 Surfaces of Constant Curvature

This section discusses surfaces of constant Gaussian Curvature, which will help us determine all such surfaces that are also surfaces of revolution. Recall that a surface of a revolution is of the form

$$M = \{(r(s)\cos\theta, r(s)\sin\theta, z(s)) : 0 \le \theta \le 2\pi,\ s \in (s_0, s_1)\}$$

where

$$\alpha(s) = (r(s), z(s))$$

is a regular unit-speed curve defined in an interval (s_0, s_1), with $r(s) > 0$; M is a surface that can be covered with two coordinate patches. Since α is unit-speed, the metric matrix is

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

and the second fundamental form has matrix

$$(L_{ij}) = \begin{pmatrix} r'z'' - z'r'' & 0 \\ 0 & rz' \end{pmatrix}.$$

The Gaussian curvature K of M can then be derived as

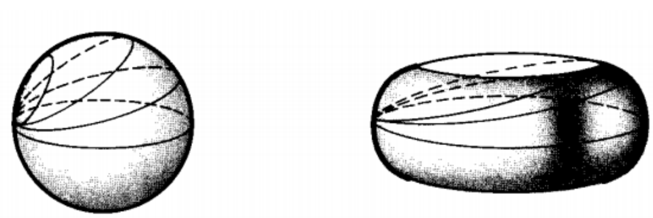
$$K = -\frac{r''}{r}.$$

If M is the surface of revolution generated by the unit speed curve α and M has constant *positive* Gaussian curvature $K = a^2 >$, then α is given by

$$r(s) = A\cos(as), \quad |s| < \pi/2a$$

$$z(s) = \pm \int_0^s \sqrt{1 - a^2A^2\sin^2(at)dt} + C.$$

If both M_1 and M_2 have the same positive Gaussian curvature, then M_1 and M_2 are locally isometric. All surfaces of revolution with constant curvature a^2 are locally isometric to the sphere of radius $1/a$, although their global properties are different; for example, on a sphere, the equator can be deformed to a point via a homotopy (see my notes for Math202A) passing through the north pole; this is not the case for the napkin ring (although the two are isometric).



If M is a surface of revolution and M has constant curvature *zero* then M is either a part of a circular cylinder, part of a plane, or part of a circular cone; all of these surfaces are locally isometric.

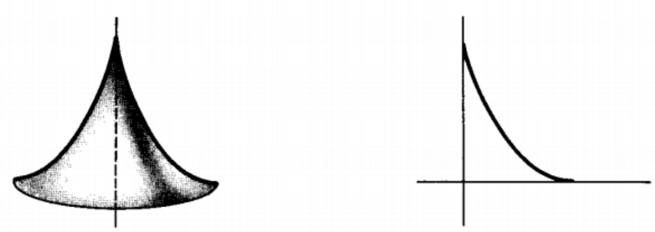
Finally, if $K = -a^2$ for $a > 0$ (constant negative Gaussian curvature) then we get one of the possibilities:

$$\begin{cases} r(s) = Ae^{as} \\ z(s) = \pm \int_0^s \sqrt{1 - a^2A^2e^{2at}}dt + D \end{cases}$$

$$\begin{cases} r(s) = B\cosh(as) \\ z(s) = \pm \int_0^s \sqrt{1 - a^2B^2\sinh^2(at)}dt + D \end{cases}$$

$$\begin{cases} r(s) = C\sinh(as) \\ z(s) = \pm_0^s \sqrt{1 - a^2C^2\cosh(at)}dt + D \end{cases}$$

In the first of these cases, the substituion $aAe^{at} = \sin\phi$, the integral describing $z(s)$ may be computed. This curve is called a **tractrix** or **drag curve**; intuitively, it is the curve formed by "dragging" a rope off the "ground" (the plane) up the z-axis. The surface of revolution generated by the tractrix is called the **pseudo-sphere** of radius $1/a$.



5 Global Theory of Space Curves

This section is a spatial interpretation of the rotational index theorem from chapter 3, and the tangent spherical image from chapter 4.

5.1 Fenchel's Theorem

The **total curvature** of a regular curve $\alpha : [0, L] \to \mathbb{R}^3$ is

$$\int_0^L \kappa ds.$$

Since $\kappa = |\mathbf{T}'|$, the total curvature is the length of the tangent spherical image

$$\mathbf{T} : [0, L] \to \mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\} = \{\mathbf{a} \in \mathbb{R}^3 || \mathbf{a}| = 1\}.$$

If $A, B \in \mathbb{S}^2$, then $\widehat{AB}$ is the distance from A to B across the shortest geodesic (the great circle through A and B). The **open hemisphere** with pole N is the set

$$\{X \in \mathbb{S}^2 : \widehat{XN} < \pi/2\}.$$

The **closed hemisphere with pole** N is the set w

$$\{X \in \mathbb{S}^2 : \widehat{XN} \leq \pi/2\}.$$

If $\widehat{AB} = \pi$ then A and B are **antipodal**. If α is a regular closed curve, its tangent spherical image does not lie in any open or closed hemisphere, unless it lies in the great circle bounding the hemisphere.

Let $\gamma(t)$ be a closed C^1 curve on $\mathbb{S}^2$. The image $\mathscr{C}$ of γ is contained in an open hemisphere of either of the following two conditions hold:

(a) The length l of γ is less than 2π; or

(b) $l = 2\pi$ but the image of γ is not the union of two great semicircles.

Fenchel's Theorem (1929)

The total curvature of a closed space curve α is at least 2π. It equals 2π if and only if α is a plane convex curve.

5.2 The Fary-Milnor Theorem

Consider a sphere with an **oriented** great circle (meaning the great circle is parameterized in a specific direction). Then each great circle corresponds to a point on the sphere (the pole of the hemisphere to the left of that great circle). This relation is bijective. The **measure** of a set of oriented great circles is the area of the corresponding subset of $\mathbb{S}^2$.

If $W \in \mathbb{S}^2$, let $W^\perp$ be the corresponding great circle. For a regular curve γ with image $\mathscr{C}$, let $n_\gamma(W)$ be the number of points in $\mathscr{C} \cap W^\perp$; note $n_\gamma(W)$ is a geometric property.

Crofton's Formula

Let $\mathscr{C}$ be the image of a regular curve $\gamma(t)$ on $\mathbb{S}^2$ of length l. The measure of the set of oriented great circles that intersect $\mathscr{C}$, counted with multiplicity, is $4l$.

Now we define

$$D^2 = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$$

is the **closed unit disk**.

$$\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 | x^1 + y^1 = 1\}$$

is the **unit circle**.

Fary-Milnor Theorem

If α is a simple knotted regular curve, then the total curvature of α is at least 4π.

5.3 Total Torsion

We have already studied $\int \kappa ds$; here we examine the (weaker) results of $\int \tau ds$. The **total torsion** $\int \tau ds$ of a closed unit-speed curve $\alpha(s)$ is zero. The converse, proved by Scherrer in 1940, is also true; if M is a surface in $\mathbb{R}^3$ with total torsion zero for *all* closed curves in M, then M is part of a plane or sphere.

6 Global Theory of Surfaces

Much like Chapters 3 and 5 elevate the concepts from Chapter 2 to global results (and are consequently much more difficult than Chapter 2), this chapter addresses the global version of Chapter 4.

6.1 Simple Results

This first section deals with some simple results relating a surface to its topology.

Compactness

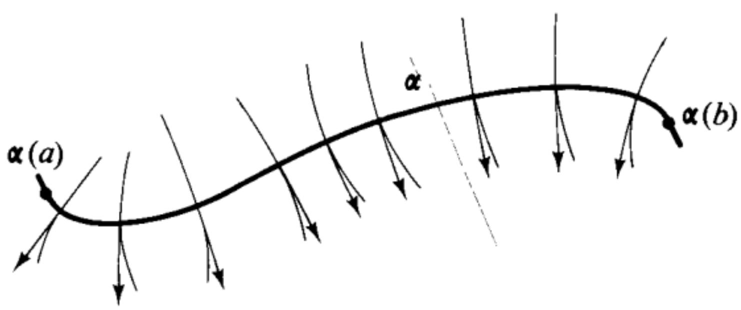
If $P \in \mathbb{R}^3$, then $|P|$ is the **distance** from P to the origin O of $\mathbb{R}^3$. Let $\bar{B}_r = \{P \in \mathbb{R}^3 | |P| < r\}$. $M \subset \mathbb{R}^3$ is **bounded** if there is an $r > 0$ such that $M \subset \bar{B}_r$. M is **closed** if for each sequence $\{P_n\}$ of points such that $P_n \in M$ and $\lim_{n \to \infty} P_n = P$ exists, P is in M. If $M \subset \mathbb{R}^3$

is both closed *and* bounded, then M is **compact**.

If M is compact, the smallest enclosing ball $\bar{B}_r$ must intersect M at some point in its boundary. As a result, every compact surface M must have positive Gaussian curvature at at least one point. **Meusnier's theorem** (1785) builds off this notion – if a surface M sphere is a compact connected surface whose points are all umbilics, then M must be a sphere.

6.2 Geodesic Coordinate Patches

If $\mathbf{x} : \mathcal{U} \to \mathbb{R}^3$ is a coordinate patch such that $g_{11} = 1$ and $g_{12} = 0$, then $\mathbf{x}$ is called a **geodesic coordinate patch**. If, in addition, there is a curve γ on M defined on $[a, b]$ such that $\gamma([a, b]) \subset \mathbf{x}(\mathcal{U})$ and such that the u^2-curve through a point of the image of γ is γ itself, then $\mathbf{x}$ is called a **geodesic coordinate patch along** γ. We can construct a geodesic coordinate patch by selecting a non-closed curve α and creating a neighborhood of geodesics perpendicular to α, so that all u^1 curves will be geodesics and the u^2 curve through a point of the image of α is α.



The first fundamental form of the geodesic coordinate patch then has matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & h^2 \end{pmatrix}$$

with $h = |\mathbf{x}_2| > 0$. In the special case where α is unit-speed, the function $h = |\mathbf{x}_2|$ has two important properties; that $h(0, u^2) = |\alpha'(u^2)| = 1$ and $\partial(h^2)/\partial u^1(0, u^2) = \alpha''(u^2)$ is normal to the surface. These properties then tell us that any two surfaces of the same constant Gaussian curvature must be locally isometric.

More specifically, for two surfaces of the same constant curvature if there are any two points P and Q in M and N with geodesics α and γ through P and Q (all respectively), then there is a local isometry f of a neighborhood $\mathcal{U}'$ of P with a neighborhood $\mathcal{V}'$ of Q such that $f(P) = Q$ and $\gamma = f \circ \alpha$. In particular, if $M = N$ there is a local isometry mapping P to Q (a "translation") and if $P = Q$ there is a local isometry sending a given geodesic through P to another given geodesic through P (a "rotation").

6.3 Orientation and Angular Variation

Let $\gamma(t)$ be a piecewise regular simple closed curve in a surface M with period L. Then suppose $\mathbf{Z}(t)$ is a continuous vector field along γ which is differentiable along the regular segments of γ. In general, $\mathbf{Z}(0) \neq \mathbf{Z}(L)$. We are interested in seeing how different these vectors are (in terms of orientation). In three dimensions, the cosine of the angle between space vectors is well defined, but the orientation is not (there is no good definition of "counterclockwise").

A surface M is **orientable** if there is a continuous function $v : M \to \mathbb{S}^2$ with $v(P)$ normal to M at P for all $P \in M$. More intuitively, we may consider a surface orientable if it has a clearly defined "outside" and "inside" – and can therefore define "counterclockwise" consistently across the surface. The sphere, cylinder, and torus are all orientable (but the Möbius strip and Klein bottle are not). A deep topological result: **every compact three-dimensional surface is orientable**. The angle between vectors $\mathbf{X}$ and $\mathbf{Y}$ is defined up to sign by the inner product, i.e. $\cos\theta = \langle \mathbf{X}, \mathbf{Y} \rangle / |\mathbf{X}||\mathbf{Y}|$. The sign of θ is determined by the triple product $[\mathbf{X}, \mathbf{Y}, \mathbf{n}]$ (equivalently, the right hand rule). We denote this angle between $\mathbf{X}$ and $\mathbf{Y}$ as $\sphericalangle(\mathbf{X}, \mathbf{Y})$.

A subset $\mathscr{R}$ of a surface M is a **region** in M if $\mathscr{R}$ is open and if any two points of $\mathscr{R}$ may be joined by a curve in $\mathscr{R}$. If $\mathscr{R}$ is a region in M, then the **boundary** of $\mathscr{R}$, $\partial\mathscr{R}$, is the set $\{P \in M | P \notin \mathscr{R},\ \exists \{P_n\} \in \mathscr{R}, \lim P_n = P\}$. A curve γ **bounds** a region $\mathscr{R}$ if the image of γ is the boundary of $\mathscr{R}$ and the intrinsic normal $\mathbf{S}$ points *into* $\mathscr{R}$ at all points of γ and likewise $-\mathbf{S}$ points out of $\mathscr{R}$.

To measure the total change in angle in $\mathbf{Z}$ along γ we need to replace the x-axis. Suppose γ bounds $\mathscr{R}$. Then it can be shown that there is a region $\mathscr{S}$ in M containing $\mathscr{R}$ and the image of γ and a field of unit vectors in $\mathscr{S}$; i.e. a differentiable assignment of unit tangent vectors $\mathbf{V}(P)$ to each point $P \in \mathscr{S}$.

Let $\alpha(t) = \sphericalangle(\mathbf{V}(\gamma(t)), \mathbf{Z}(t))$ with α continuous, and hence differentiable. Once $\mathbf{V}$ is fixed, α is unique up to an integral multiple of 2π so $d\alpha/dt$ is a function in a single variable ($\mathbf{Z}$). The **total angular variation** of $\mathbf{Z}$ along γ with respect to $\mathbf{V}$ is

$$\delta_{\mathbf{V}}\alpha = \int_0^L \frac{d\alpha}{dt}dt.$$

In general, $\delta_{\mathbf{V}}\alpha$ depends only on $\mathbf{V}$; however, if we can continuously shrink γ to a single point in $\mathscr{R}$, we can remove this dependency.

Null-homotopic Map

Let γ be a closed curve which bounds a region $\mathscr{R}$. Let σ be any closed curve of period L which is either γ or lies in $\mathscr{R}$. Let $\sigma(0) = \mathbf{x}_0$. σ is **null-homotopic** (it is homotopic to a constant function) in $\mathscr{R}$ if for each $s \in [0, 1]$ there is a closed curve σ_s in M such that

(a) $\sigma_s(0) = \mathbf{x}_0$;

(b) $\sigma_0(t) = \sigma(t)$ and $\sigma_1(t) = \mathbf{x}_0$ (so σ_1 is the constant curve);

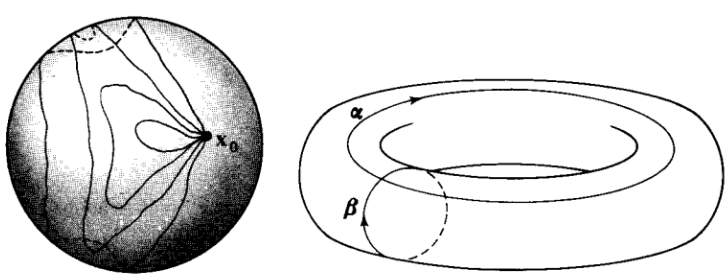
(c) $\sigma_s(t) \in \mathscr{R}$ for all $0 < s \le 1$ and $t \in (0, L)$;

(d) The function $\Gamma : [0, L] \times [0, 1] \to M$ given by $\Gamma(t, s) = \sigma_s(t)$ is continuous.

If γ is null-homotopic in $\mathscr{R}$ then the total angular variation is independent of $\mathbf{V}$.

6.4 The Gauss-Bonnet Formula

A region of a surface is **simply connected** if every closed curve in that region is null-homotopic. The sphere is simply connected (every closed curve on the sphere can be shrunk continuously to a point); the torus is not.



Gauss-Bonnet Formula

Let γ be a piecewise regular curve contained within a simply connected geodesic coordinate patch and bounding a region $\mathscr{R}$ in the patch. Let the jump angles at the junctions be $\alpha_1, ..., \alpha_n$. Then

$$\iint_{\mathscr{R}} K dA + \oint_\gamma \kappa_g ds + \sum \alpha_i = 2\pi.$$

As an example, take the unit sphere $\mathbb{S}^2$. Let γ be the triangle whose sides are geodesics and whose interior angles are $\beta_1, \beta_2, \beta_3$. Then $\alpha_i = \pi - \beta_i$ (interior angles are jump angles).

$$2\pi = \iint_{\mathscr{R}} K dA + \oint_\gamma \kappa_g ds + \sum \alpha_i = \text{area}(\mathscr{R}) + 0 + 3\pi - \sum \beta_i.$$

The quantity $\sum \beta_i - \pi$ equals the area of $\mathscr{R}$ in spherical geometry (Legendre), and is called the **angular excess** of γ.

6.5 Euler Characteristic

A **polygon** on a surface M is a piecewise regular curve γ whose segments are geodesics and which bounds a simply connected region $\mathscr{R}$. Let M be compact. Suppose M can be broken into regions bounded by polygons, each region in a simply connected geodesic coordinate patch. If V is the number of vertices, E edges, and F faces, the **Euler characteristic** of M with respect to this particular decomposition is

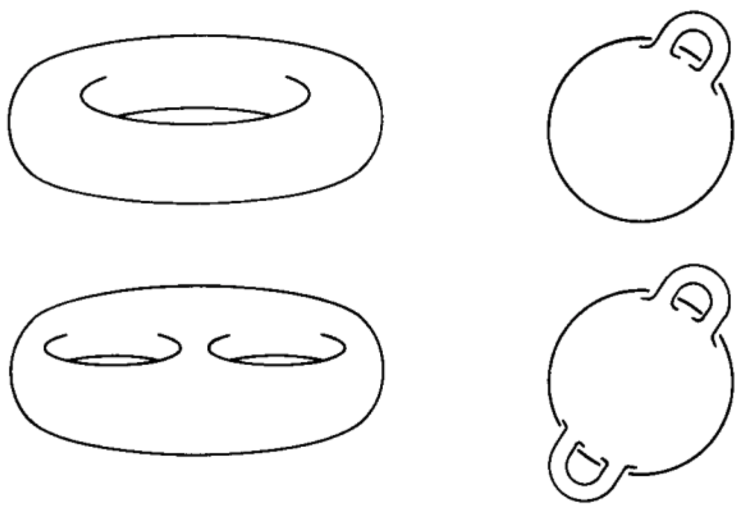
$$\chi = F - E + V.$$

The Gauss-Bonnet formula can then be restated, for compact M, as

$$\iint_M K dA = 2\pi\chi.$$

The quantity on the LHS is called the **curvatura integra** or **total curvature** of the surface. χ is well-defined, so it must be independent of how the polygons are chosen, so long as each is small enough to be contained in a simply connected geodesic coordinate patch. Since χ is an integer, we notice that $(1/2\pi) \iint_M K dA$ must be an integer as well (without *a priori* reason). We choose geodesic polygons here so the $\oint_\gamma \kappa_g ds$ term disappears from the Gauss-Bonnet formula.

χ has been defined combinatorially; it is an intrinsic geometric invariant; we can also define it topologically. Every compact surface in $\mathbb{R}^3$ "looks like" a sphere with handles. The number of handles is the **geometric genus** of M and is denoted g:



If M is a compact surface in $\mathbb{R}^3$, then $\chi = 2(1-g)$. The Euler characteristic of a compact surface in $\mathbb{R}^3$ is even and at most 2; we can then generalize the Gauss-Bonnet theorem for this particular context.

Let γ be a piecewise regular curve in an oriented surface M. Suppose γ bounds $\mathscr{R}$; then

$$\iint_{\mathscr{R}} K dA + \oint_{\gamma} \kappa_g ds + \sum (\pi - \alpha_i) = 2\pi\chi(\mathscr{R}),$$

where α_i are the interior angles of γ and χ is the Euler characteristic of $\mathscr{R}$ found by splitting $\mathscr{R}$ into polygons and counting those edges and vertices lying on γ in addition to those in $\mathscr{R}$.

6.6 Theorems of Jacobi and Hadamard

Here we examine two more theorems in global differential geometry; their proofs are applications of the Gauss-Bonnet theorem. Recall that for a regular space curve γ of positive curvature, the normal spherical image σ is the curve $\sigma(s) = \mathbf{N}(s)$ where $\mathbf{N}$ is normal to γ.

Jacobi's Theorem (1842)

Let γ be a regular closed unit-speed space curve with positive curvature. Assume that σ, its normal spherical image, is a simple curve. Then it divides the unit sphere into two regions of equal area.

Hadamard's Theorem (1897)

Let M be a compact surface in $\mathbb{R}^3$. If $K > 0$ everywhere, then the surface is necessarily convex.

6.7 Index of a Vector Field

Let M be a compact surface with a vector field $\mathbf{V}$. $P \in M$ is an **isolated zero** of $\mathbf{V}$ if there is an open set $\mathcal{U}$ about P with P the only point in $\mathcal{U}$ where $\mathbf{V}$ is zero. Suppose P is isolated and let γ be a simple closed piecewise regular curve bounding a simply connected region $\mathscr{R}$ such that P is the only zero in $\mathscr{R}$. The **index** of $\mathbf{V}$ at P is

$$i_P(\mathbf{V}) = \frac{1}{2\pi}\delta\sphericalangle(\mathbf{U}, \mathbf{V})$$

where $\mathbf{U}$ is any field of unit vectors in $\mathscr{R}$. If M is compact with a vector field $\mathbf{V}$ with finitely many zeros, then the **total index** of $\mathbf{V}$ is $I(\mathbf{V}) = \sum i_P(\mathbf{V})$.

Poincaré-Brouwer Theorem

If M is a compact surface and $\mathbf{V}$ is a vector field on M with finitely many zeros, then $I(\mathbf{V}) = \chi(M)$, the Euler characteristic of M. This implicitly implies that any vector field on $\mathbb{S}^2$ must have a zero; this is also known as the **hairy ball theorem**.

7 Introduction to Differential Manifolds

Just as in Chapter 4 we introduced the ideas of "space" (in the form of the surface) and "line" (in the form of the geodesic), here we take a more abstract view generalizing to Euclidean n-space. However, this is slightly different from our previous treatment; we previously treated surface as the 3-D image of a function on a 2-D domain; however, a manifold need not lie in the next-dimensional Euclidean space (they don't need to be in Euclidean space at all).

7.1 Analytic Preliminaries

We begin with a quick review of some tools from analysis. Let X be a set. A **metric** d on X is an assignment of an nonnegative number $d(x, y)$ to each pair of points $x, y \in X$ such that:

(a) $d(x, y) = 0$ if and only if $x = y$;

(b) $d(x, y) = d(y, x)$;

(c) if $z \in X$ then $d(x, y) \leq d(x, z) + d(z, y)$.

A **metric space** is a space with a metric d.

If $x \in X$ and $\varepsilon > 0$ then the **open ball of radius** ε **about** x is

$$B_\varepsilon(x) = \{y \in X | d(x, y) < \varepsilon\}.$$

$S \subset X$ is **open in X** if for each $x \in S$ there is an $\varepsilon > 0$ such that $B_\varepsilon(x) \subset S$. If S is open and $x \in S$, then S is a **neighborhood of** x. The finite intersection of open sets is open, and the union of open sets is open. For any two nonidentical points in X, there are two non-overlapping open sets in X such that one point is in one set and the other point is in the other. If f maps the metric space X into the metric space Y, then f is **continuous** if every open set in Y has preimage (under f) in X.

We now review some basic calculus. If f maps $\mathbb{R}^n$ to $\mathbb{R}^n$, we may represent f as a set of m functions of n variables. If f is differentiable then the **Jacobian** at each point $p \in \mathbb{R}^n$ is:

$$J_f(p) = \begin{pmatrix} \frac{\partial f^1}{\partial x^1}(p) & \dots & \frac{\partial f^1}{\partial x^n}(p) \\ \frac{\partial f^2}{\partial x^1}(p) & \dots & \frac{\partial f^2}{\partial x^n}(p) \\ & \ddots & \\ \frac{\partial f^m}{x^1}(p) & \dots & \frac{\partial f^m}{x^n}(p) \end{pmatrix}.$$

Diffeomorphism

Suppose U is open in $\mathbb{R}^n$, V is open in $\mathbb{R}^n$, and $f : U \to V$. f is a C^k **diffeomorphism** if f is C^k and there is a C^k function $g : V \to U$ such that $f \circ g$ is the identity map on V. g is the **inverse** of f.

The **inverse function theorem** states that if U is an open set in $\mathbb{R}^n$, $p \in U$, and $f : U \to \mathbb{R}^n$, and if $J_f(p) \neq 0$, then there are neighborhoods N_p of p and $N_{\varphi(p)}$ of $\varphi(p)$ such that

$$f|_{N_p} : N_p \to N_{\varphi(p)}$$

is a diffeomorphism.

The **implicit function theorem** states that, if $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a C^k function, $a \in \mathbb{R}^{n+1}$, and $(\partial f/\partial u^i)(a) \neq 0$, then there is a neighborhood W of a in $\mathbb{R}^{n+1}$ and a C^k function $g : \hat{W} \to \mathbb{R}$ such that for $n+1$-dimensional vector $\mathbf{w}$, $f(\mathbf{w}) = 0$ if and only if $w^i = g(\hat{w})$.

7.2 Manifolds

Here we offer the definitions crucial to this chapter – that of the n-manifold. It will be clear that "surfaces" as we have discussed are those 2-manifolds which are embedded in $\mathbb{R}^3$. An n-manifold looks (locally) like $\mathbb{R}^n$.

Let M be a metric space and $p \in M$. A **coordinate chart about** p **of dimension** n is a neighborhood U of p and an injective continuous function $\varphi : U \to \mathbb{R}^n$ such that $\tilde{U} = \varphi(U)$ is open in $\mathbb{R}^n$. (U, φ) is a **proper coordinate chart** if $\varphi^{-1} : \varphi(U) \to U \subset M$ is continuous.

n-Manifolds

Let M be a metric space. M is an $n-$**dimensional** C^∞ **manifold** if there is a collection $\mathscr{A}$ of coordinate charts (U, φ), called the **atlas** of M, such that:

(a) for each $p \in M$ there is a proper coordinate chart $(U, \varphi) \in \mathscr{A}$ of dimension n with $p \in U$;

(b) if (U, φ), $(V, \psi) \in \mathscr{A}$ with $U \cap V \neq \emptyset$ then

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

is a C^∞ diffeomorphism of open sets of $\mathbb{R}^n$;

(c) $\mathscr{A}$ is maximal with respect to conditions (a) and (b), i.e. $\mathscr{A}$ contains all possible charts with the above properties.

This is essentially the same as the definition of a surface, except the direction of the maps is reversed and U is a subset of M instead of being a subset of Euclidean space.

If $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is C^∞ then the **gradient** of f is the function $\nabla f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ given by

$$(\nabla f)(p) = \left(\frac{\partial f}{\partial u^1}(p), ..., \frac{\partial f}{\partial u^{n+1}}(p) \right).$$

Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a C^∞ function and

$$M_f = \{x \in \mathbb{R}^{n+1} | f(x) = 0\}.$$

If $(\nabla f)(p) \neq 0$ for all $p \in M_f$ then M_f is a C^∞ n-manifold called the **hypersurface defined by** f.

7.3 Lie Groups and Tangent Bundles

We would like to define an analogous concept to the tangent vector for our n-manifolds. This analogy is non-trivial; for a surface in $\mathbb{R}^3$, we may define a tangent vector by our well-defined notion of a derivative in $\mathbb{R}^3$. However, for a manifold M, there is not necessarily any ambient Euclidean space, and it does not make sense to differentiate a metric space.

In calculus, a vector $\mathbf{v}$ at a point may be viewed as a directional derivative, as in $\mathbf{v}(f) = \sum a^i \frac{\partial f}{\partial u^i}(p)$. We then define a tangent vector as a real-valued operator on the set of differentiable functions on M obeying the properties of a derivative. Let $p \in M$ and let $f : M \to \mathbb{R}$. f is **differentiable of class C^l at** p if there is a proper coordinate chart (U, φ) about p such that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ is of class C^l at $\varphi(p)$. If f maps two C^∞ manifolds, such that (U, φ) is a chart about p and (V, ψ) is a chart about $f(p)$, and $\phi \circ f \circ \varphi^{-1}$ is of class C^l at $\varphi(p)$, then f is **differentiable**. The set $\mathfrak{F}(M) = \{f : M \to \mathbb{R} | f$ is of class C^∞ is the set of all C^∞ functions on a manifold.

Lie Group

If G is a manifold which is also a group such that both $\mu : G \times G \to G$ by $\mu(x, y) = xy$ and $\iota : G \to G$ by $\iota(x) = x^{-1}$ are of class C^∞, then G is known as a **Lie group**.

If W is an open set in M, with $p \in W$, then there exists an open set V with $p \in V \subset W$ with a C^∞ function $f : M \to \mathbb{R}$ where $f(W^c) = 0$, $f(V) = 1$, and $0 \leq f(M) \leq 1$. From this, we say that if W is a neighborhood about

p and $F : W \to \mathbb{R}$ is differentiable, then there exists a differentiable set $\tilde{F} : M \to \mathbb{R}$ which agrees with F on an open subset V of W. If you have taken complex analysis, this is functionally equivalent to analytic continuation.

The **tangent vector to a manifold** M **at a point** p is a function $X_p : \mathfrak{F}(M) \to \mathbb{R}$, whose value at f is denoted $X_p(f)$, such that for all $f, g \in \mathfrak{F}$ and $r \in \mathbb{R}$:

(a) $X_p(f+g) = X_p(f) + X_p(g)$;

(b) $X_p(rf) = rX_p(f)$;

(c) $X_p(fg) = f(p)X_p(g) + g(p)X_p(f)$.

Let (U, φ) be a chart about $p \in M$ and let $u^1, u^2, ...$ be Cartesian coordinates in $\mathbb{R}^n$. Then $(\partial/\partial x^i)_p$ is the tangent vector given by

$$\left(\frac{\partial}{\partial x^i}\right)_p (f) = \frac{\partial(f \circ \varphi^{-1})}{\partial u^i}(\varphi(p)).$$

The value of this derivative depends entirely on the chart used. The **tangent space** to M at p, T_pM, is the set of all tangent vectors to M at p. The disjoint union of all tangent spaces of M,

$$TM = \bigsqcup_{p \in M} T_pM,$$

is known as the **tangent bundle of** M.

7.4 Lie Brackets

The **field of vectors** X is an assignment of a tangent vector $X_p \in T_pM$ to each $p \in M$. If X is a field of vectors and $f \in \mathfrak{F}(M)$, then we may define a real-valued function Xf on M by $(Xf)(p) = X_pf$. If $Xf \in \mathfrak{F}(M)$ for each $f \in \mathfrak{F}(M)$, then X is called a **vector field**. The collection of all vector fields on a surfaces is denoted as $\mathfrak{X}(M)$.

Lie Bracket

If $X, Y \in \mathfrak{X}(M)$, then the **Lie bracket** of X and Y, $[X, Y]$, is the field of vectors defined by

$$[X, Y]_pf = X_p(Yf) - Y_p(Xf)$$

for $f \in \mathfrak{F}(M)$ and $p \in M$. $[X, Y]$ is a vector field on M.

If $X, Y, Z \in \mathfrak{X}(M)$ and $r \in \mathbb{R}$, then

(a) $[X, Y] = -[Y, X]$ and $[rX, Y] = r[X, Y]$;

(b) $[X + Y, Z] = [X, Z] + [Y, Z]$, $[Z, X + Y] = [Z, X] + [Z, Y]$;

(c) (**Jacobi's Identity**) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$.

7.5 Differential of a Map

If $\Phi : M \to N$ is differentiable, the **differential of** Φ **at** p is the function $(\Phi_*)_p : T_pM \to T_{\Phi(p)}N$ defined by

$$(\Phi_*)_p(X_p)(f) = X_p(f \circ \Phi).$$

The differential is well-defined, i.e.

$$\Phi : M \to N,\ X_p \in T_pM \implies (\Phi_p)_*(X_p) \in T_{\Phi_{(p)}}N.$$

If $\Phi : M \to N$ and $\Psi : N \to P$ are differentiable maps of the manifolds M, N, P then the differential of the composition of Φ and Ψ is equal to the composition of their individual differentials:

$$(\Phi \circ \Psi)_* = (\Phi)_* \circ (\Psi)_*.$$

If M^m and N^n are manifolds, then M is a **submanifold** of N if there is a differentiable function $\Phi : M \to N$ such that both Φ and $(\Phi_*)_p$ are injective for all p; Φ is then called an **embedding** of M in N. The dimensionality of M is at most that of N.

If $M = M_f$ is the hypersurface defined by $f : \mathbb{R}^{n+1} \to \mathbb{R}$, then T_pM is isomorphic (as a vector space) to

$$(\Phi_*)_pT_pM = \{X_p \in T_p\mathbb{R}^{n+1} | \langle \nabla f_p, X_p \rangle = 0\}$$

where Φ is the inclusion of M in $\mathbb{R}^{n+1}$. The concept of embeddings is a deep and complex subject and is better left to a formal examination of differential topology; however we do present the following theorem:

Whitney Embedding Theorem

Every n-manifold embeds in $\mathbb{R}^{n+1}$.

7.6 Linear Connections

Recall that we were able to define a parallel vector field to a surface by differentiating the vector field along a curve in the direction of its tangent, and from there arrived at the definition of a geodesic (a curve along which the tangent vector field is parallel). This section explores the generalization of differentiating a vector field with respect to a vector field by introducing the **linear connection**.

A **linear connection** (or just a **connection**) on M is a linear function $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ (denoted $\nabla_X Y$) such that

$$\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z, \quad \nabla_{fX} Y = f\nabla_X Y, \quad \nabla_X fY = (Xf)Y + f\nabla_X Y.$$

If ∇ is a connection on M and (U, φ) is a proper coordinate chart, then the **Christoffel symbols** of ∇ with respect to (U, φ) are the functions $\Gamma^k_{ij} \in \mathfrak{F}(U)$:

$$\nabla_{\partial/\partial x^i}\left(\frac{\nabla}{\nabla x^j}\right) = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x^k}.$$

Note that unlike before, this formulation is *not* symmetric.

Z is a **vector field along** $\alpha : I \to M$ if Z assigns to each $t \in I$ an element $Z_{\alpha(t)} \in T_{\alpha(t)}M$ such that $t \to Z_{\alpha(t)}(f)$ is a differentiable real-valued function of t for each $f \in \mathfrak{F}(M)$. The **tangent vector field** T_α is given by $(T_\alpha)_{\alpha(t)} = (\alpha_*)_t(d/dt)$.

Let $\alpha : I \to M$ and $Y \in \mathfrak{X}(M)$. Let t_0 be given and let X be any vector field on M such that $X_{\alpha(t_0)} = T_{\alpha(t_0)}$. $\nabla_T Y$, the **covariant derivative of Y with respect to α**, is the vector field defined by

$$(\nabla_T Y)_{\alpha(t_0)} = (\nabla_X Y)_{\alpha(t_0)}.$$

7.7 Riemannian Metrics

Riemannian Manifolds

A **field of metrics** g on a manifold M is an assignment of a linear map $g_p : T_pM \times T_pM \to \mathbb{R}$ to each $p \in M$ such that for all X_p, Y_p, $Z_p \in T_pM$ and $r \in \mathbb{R}$:

(a) $g_p(X_p, Y_p) = g_p(Y_p, X_p)$

(b) $g_p(X_p, X_p) \geq 0$ with $g_p(X_p, X_p) = 0$ if and only if $X_p = 0$.

A **Riemannian metric** on a manifold M is a field of metrics g such that $g(X, Y) \in \mathfrak{F}(M)$ for all $X, Y \in \mathfrak{X}(M)$. A **Riemannian manifold** is a manifold with a fixed Riemannian metric.

If $A : T_pM \to T_qM$ is a linear transformation, it is an **isometry** if for all $X_p, Y_p \in T_pM$, $g_p(X_p, Y_p) = g_q(AX_q, AX_p)$. A linear connection ∇ on a Riemannian manifold is **metrical** if for all $X, Y, Z \in \mathfrak{X}(M)$:

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Note the similarity to the derivative of the dot product.

A linear connection ∇ is **torsion-free** if

$$\nabla_X Y - \nabla_Y X = [X, Y] \text{ for all } X, Y \in \mathfrak{X}(M).$$

This occurs precisely when the Christoffel symbols form a symmetric matrix. The **Fundamental Lemma of Riemannian Geometry** states that for every Riemannian manifold (M, g) there is a unique torsion-free metrical linear connection ∇ on M; this connection is known as the **Riemannian connection**.

Hilber's Theorem and Converse

If a connected Riemannian manifold M is complete as a metric space under the metric induced by g, then any two points can be joined by a geodesic which minimizes the distance between the points, where the distance is given by

$$d(p, q) = \inf |\alpha|, \quad |\alpha| = \int_a^b \sqrt{g_{\alpha(t)}(T_{\alpha(t)}, T_{\alpha(t)}} dt.$$

The partial converse, the **Hopf-Rinow theorem**, states that if every geodesic on M may be extended indefinitely, then any two points may be joined by a geodesic of minimal length and M is complete as a metric space under the metric determined by g.

Gauss's *Theorema Egrigium* gives us insight as to how to define the concept of *curvature* on an arbitrary Riemannian manifold. The **Riemann-Christoffel curvature tensor of type** $(1,3)$ is the map $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

A surface with $R \equiv 0$ is locally indistinguishable from $\mathbb{R}^n$ geometrically.

Ricci and Scalar Curvature

If $X_p, Y_p \in T_pM$, we may define $\Xi_p(X_p, Y_p) : T_pM \to T_pM$ by

$$\Xi_p(X_p, Y_p)V_p = R(V_p, X_p)Y_p.$$

The **Ricci curvature tensor** S is an assignment to each $p \in M$ of a function $S_p : T_pM \times T_pM \to \mathbb{R}$ defined by

$$S_p(X_p, Y_p) = \mathbf{tr}(\Xi_p(X_p, Y_p)).$$

If M^n is a Riemannian manifold with Ricci tensor S, then the **scalar curvature** of M at p is

$$\sum_{i=1}^{n} S_p((X_i)_p, (X_i)_p)$$

where $\{X_i\}$ is an orthonormal basis for T_pM. These two concepts are widely applicable in the study of relativity in physics (although it should be noted that in the context of relativity this is *not* Riemannian geometry, as in physics we require an exclusively *nondegenerate* (positive definite) metric as opposed to our general Riemannian one).